Constructing and Representing a Quantitative Structure: A Conceptual Analysis

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Abstract: In this paper, we offer a conceptual model of mental actions involved in constructing a quantitative structure by reasoning with amounts of change in the context of constructing and interpreting graphs. This conceptual model provides a perspective on how students can engage in quantitative and algebraic reasoning when identifying an invariant relationship between two quantities' magnitudes and developing productive meanings for graphs. We conclude with implications including discussing that fostering students' reasoning with amounts of change in quantities' magnitudes have the potential for improving instruction, especially for younger students, on constructing and interpreting graphs and linear and non-linear relationships.

Introduction

Researchers have emphasized the importance of students conceiving situations as composed of quantities whose values or magnitudes vary as a foundation to their developing understanding of critical topics in mathematics, such as rate of change (Johnson, 2015a), non-linear relationships (Castillo-Garsow, 2010; Ellis, 2011) and function (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Providing students opportunities to construct quantities and their covariation in a situation also helps them productively use mathematical representations (e.g., equations and graphs) because the construction and interpretation of representations can emerge from reasoning about quantitative relationships (Frank, 2017; Moore & Thompson, 2015; Thompson, 2011).

Thompson and Carlson (2017) argued that a researcher who claims someone has conceived of a quantitative structure must describe how this person constructed quantities and relationships among them, "...otherwise the claim is vague" (p. 444). In this paper, in order to describe the process of constructing a quantitative structure as to not be "vague", we present a conceptual analysis that articulates the cognitive operations involved in reasoning with relationships between quantities in dynamic situations. We specifically provide an explanation of a particular quantitative structure when determining an invariant covariational relationship between two quantities that vary in a situation. This elaboration incorporates central constructs of quantitative (Thompson, 1990) and core practices of algebraic reasoning (Stephens, Ellis, Blanton, & Brizuela, 2017; Smith III & Thompson, 2008).

Background

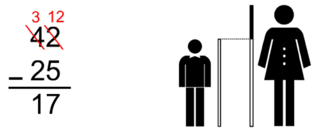
Quantitative and covariational reasoning

Aligning with our theoretical perspective (i.e., von Glasersfeld's [1995] radical constructivism), we situate a student's learning in terms of *her* or *his* conceiving and reasoning about measurable attributes of objects (i.e., quantities). Thompson (1990, 2011) defined reasoning about measurable attributes as quantitative reasoning, which involves someone's reasoning about a situation by constructing quantities, quantitative operations, and relationships between conceived quantities. By *quantitative operation*, we mean the conception of producing a new quantity from two others, and by *quantitative relationships*, we mean the conception of these three quantities so that they exist simultaneously and in a relationship with each other (Thompson, 1990, 2011). That is, while quantitative operation has to do with the *operation* in which an individual creates a new quantity, a quantitative relationship has to do with *relating* the resultant quantity with its operands. *Operands* are two quantities that are known and being considered in operating, and *the resultant quantity* is the new quantity that becomes known—including its type and unit—as a result from operating (Thompson, 1990). Then, these conceptions constitute a *quantitative structure*, which is defined as network of quantitative relationships (Thompson, 1990).

Thompson and Carlson (2017) emphasized the role conceiving a quantitative structure plays as a foundation for students' (co)variational reasoning (i.e., attending to how one quantity varies in relation to the other). Thompson and Carlson explained that students may initially construct a quantitative structure by statically envisioning quantities (i.e., not varying) in a situation. Then, as long as a student envisions a varying quantity in that structure, the student can envision other quantities varying according to the quantitative relationships he or she has constructed with that varying quantity. For example, in order to determine a pattern of *differences* in a quantity's variation in relation to the other, a student can coordinate the variation of two quantities' values or magnitudes and the variation of the resultant *difference* quantity's values or magnitudes (e.g., as two quantities increase, the difference of these quantities decrease).

Carlson et al. (2002) provided a covariational reasoning framework in order to describe the mental actions involved in applying covariational reasoning when interpreting and representing dynamic events and functions. In their framework, Carlson et al. specified five mental actions involved in coordinating quantitates that range from coordinating the value of one quantity with changes in the other (Mental Action 1 in their framework) to coordinating the instantaneous rate of change of the function (Mental Action 5). In this study, we consider Mental Action 3 (MA3) to characterize mental actions of students' coordination of amounts of change of one quantity with respect to uniform changes in another. As we illustrate in more detail below, such reasoning is an integral part of determining and generalizing an invariant relationship between two quantities within and across different representations of quantities' covariation (e.g., understanding a situation, as well as constructing and interpreting a graph and its curvature).

Numerous researchers (e.g., Carlson et al., 2002; Confrey & Smith, 1995; Ellis, 2007, 2011; Johnson, 2012, 2015b; Monk & Nemirovsky, 1994) have argued for the importance of reasoning about amounts of change in one quantity in relation to uniform changes in another quantity (i.e., MA3). However, these arguments mostly included students' reasoning with numerical values. In this paper, we expand this body of literature by articulating students' reasoning about amounts of change in dynamic events and in graphs in terms of their reasoning with quantities' *magnitudes* independent of numerical *values* (see Liang, Stevens, Tasova, & Moore, 2018; Thompson, Carlson, Byerley, & Hatfield, 2014, for a detailed discussion on magnitude reasoning). Relatedly, we elaborate students' *quantitative operations* (as described above) instead of *arithmetic operations* (i.e., the actions and calculations used to evaluate or determine a quantity's value; Thompson, 1990). For example, to evaluate a difference of two quantities' values, a student might perform the arithmetic operation of subtraction in order to calculate how much one quantity's value exceeds (or falls short of) the other quantity's value (see Figure 1a). On the other hand, to "evaluate" a difference of two quantities' magnitudes, the student may engage in a process of *gross* quantification (i.e., quantitative operation of gross additive comparison; Thompson, 1990; Steffe, 1991). Such a process involves comparing one quantity's magnitude with the other quantity's magnitude and constructing a magnitude associated with that comparison (see Figure 1b), which does not require an arithmetic operation to evaluate the resultant quantity. Thompson (1990) claimed that a person who conceives a quantity that is generated by gross quantification will have similar cognitive activities as an expert problem solver would use to solve an applied algebra problem. Therefore, engaging with quantities' magnitudes can provide productive foundations for someone attempting to comprehend a situation in terms of quantities and relationships.



(a) (b)

Figure 1. (a) Subtraction algorithm (b) An image of an additive comparison based in magnitudes.

Algebraic Reasoning and Generalization

According to Kaput (2008), there are two core aspects of algebraic thinking. One of them is *generalizing* and representing generalizations in conventional symbolic systems. The other one is about actions on those generalizations by reasoning with linguistic elements. Building on Kaput's work, Blanton, Levi, Crites, and Dougherty (2011) provided four aspects of algebraic reasoning: *generalizing*, *representing*, *justifying*, and *reasoning with* mathematical structure and relationships. Ellis (2011) and Stephens et al. (2017) defined generalizing as a construct in which individuals engage in at least one of the following three actions:

> (a) identifying commonality across cases (Dreyfus, 1991), (b) extending one's reasoning beyond the range in which it originated (Carraher, Martinez, & Schliemann, 2008; Harel & Tall, 1991; Radford, 2006), or (c) deriving broader results from particular cases (Kaput, 1999). (p. 387–388)

Ponte (1984) identified that the study of Cartesian graphs and functional relationships requires "identify[ing] regularities in variation in variation" (p. 130–131). In this paper, we illustrate how such reasoning can occur in order to make sense the variation in variation (by reasoning with amounts of change) in one quantity with respect to other presented in a situation and how this change can be represented in graphs. We thus describe a conceptual model of mental actions of someone who can *generalize* an invariant relationship between two quantities and *represents* this relationship in a graphical representational system. Furthermore, he or she *justifies* their actions based on the foundation of *reasoning with* quantities and their relationships.

Method

In this paper, we present a conceptual analysis (i.e., an analysis of mental operations; Thompson, 2008) of students' conceptualization of a situation as a quantitative structure. Conceptual analysis is a method by which teachers/researchers produce hypothetical models of students' knowledge in order to explain their observations of students' activities (Steffe, von Glasersfeld, Richards & Cobb, 1983). By developing a conceptual analysis, teachers/researchers can specify what mental operations are required to comprehend a particular set of concepts (von Glasersfeld, 1995). Furthermore, through situating it against the backdrop of generalizations across studies with numerous students, conceptual analysis can be used to describe propitious and coherent ways of knowing and constructing a mathematical idea (Thompson, 2008).

The conceptual analysis we provide emerged from our work with numerous students and modeling their thinking, and thus the conceptual analysis represents our own generalizations of that work (Liang & Moore, 2017; Moore, 2014, 2016; Moore & Silverman, 2015; Tasova & Moore; 2018; Tasova, Liang & Moore, 2019). We illustrate our conceptual model by using the Ferris wheel situation (see Figure 2) including a dynamic image of a rider (i.e., green bucket) who travels at a constant speed counterclockwise starting from the 3 o'clock position (Desmos, 2014). A prompt asks for a description of the covariational relationship between the height of the rider above the horizontal diameter of the Ferris wheel and arc length it has traveled for the first quarter of a rotation.

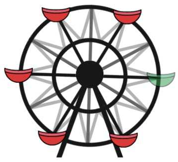


Figure 2. Ferris wheel.

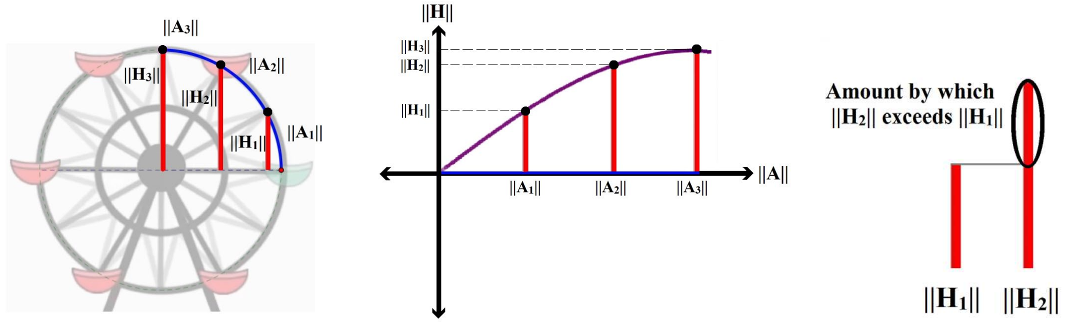
Conceptual analysis

Said generally, someone who engages in MA3 can verbally describe the relationship between the height and arc length as follows: "the height above the horizontal diameter increases by decreasing amounts as the arc length increases in equal amounts." A person's associated mental actions could involve the coordination of the relative change in magnitudes of height and arc length in order to recognize the relationship in this way (see Figure 5). In particular, the person could partition the accumulated magnitudes of arc length over the interval from 0 to pi/2 into smaller intervals of fixed magnitudes while considering the amount of change in the accumulated magnitude of the height for each successive arc length magnitude. Then, the person could recognize that amounts of change in height decrease as the arc length increases uniformly in the first quarter of rotation. In this paper, we articulate how a person may engage in such a quantitative covariational reasoning with an additional detail about the process in which one conceptualizes a quantitative structure.

Following Smith III and Thompson (2008), we identify three phases to explain the process in which one can construct a quantitative structure. Depending on the extent and nature of students' quantitative reasoning in each phase, the student may perform a different type of generalizing activity in conceptualizing the quantitative structure. Thus, we believe identifying the phases can provide an additional lens for researchers, as well as an additional structure for discussing different implications depending on the substance of the phase (see the implication section for more about the implication of use of phases). Note that these phases are to show the process in order; however, they may not occur strictly linearly in construction. We also provide Table 1 to identify the notations we use in the remainder of this paper. We use the magnitude symbol notation (e.g., $||A||$ and $||H||$) in this paper in order to refer to the magnitude that is perceptually and/or mentally available to students. It does not mean that the student uses these notations when engaging in the task.

Table 1: Notations and their descriptions

Notations	Description
$\|\|H_n\|\|$ and $\|\|A_n\|\|$	Accumulated magnitudes of height and arc length, respectively; *n* indicates the different states (e.g., $n = 0$ indicates that the magnitudes of the height and arc length are evaluated at 3:00 position in Ferris wheel, $n = 3$ indicates that the magnitudes of the height and arc length are evaluated at 12:00 position in Figure 3).
$\Delta\|\|H_n\|\|$ and $\Delta\|\|A_n\|\|$	The magnitudes of amounts of change in height and arc length, respectively (e.g., $\Delta\|\|H_1\|\|$ is the magnitude difference between $\|\|H_1\|\|$ and $\|\|H_0\|\|$).



(a) (b)

Figure 3. (a) Accumulated magnitudes of height and arc length. $||A||$ is accumulating in equal increments. (b) Difference of two quantities' magnitudes.

Phase 1: Basic

In phase 1, the person begins with constructing equal arc lengths in order to investigate how height changes in relation to arc length (see Liang & Moore, 2017, for more details of partitioning activity). The person conceives the magnitude of arc length accumulating in equal increments (i.e., $||A_n||$ and $||A_{n+1}||$ where $\Delta||A_n||=\Delta||A_{n+1}||$) as the rider travels. The person then constructs magnitudes of height (i.e., $||H_n||$, see red segments in Figure 3) corresponding to each equal increment of arc length (i.e., $||A_n||$). Then, the person determines that $||H_{n+1}||$ is greater than $||H_n||$ and, in turn, conceives the quantitative *difference* (i.e., change in height's magnitude) between two magnitudes of height corresponding to one increment of change in the magnitude of arc length as the rider travels. Note that the person need not know neither the specific value of any of the differences nor the specific values of the accumulated heights of the rider to think about the difference as a quantity. But, the person might anticipate a numerical comparison of $||H_{n+1}||$ and $||H_n||$. Figure 3b shows the magnitude of this quantitative difference between $||H_{n+1}||$ and $||H_n||$ where $n = 1$. Here, the quantitative operation that the person engages is gross additive comparisons (Steffe, 1991) among the accumulated height magnitudes at successive states (i.e., $||H_0||$, $||H_1||$, $||H_2||$, and $||H_3||$), which results in a new quantity that is the amount of change magnitude in height (i.e., $\Delta||H_1||$, $\Delta||H_2||$, and $\Delta||H_3||$, see green segments in Figure 5a). Smith III and Thompson (2008) refer to this action as the basic case because the difference, in this case, compares two states (i.e., $||H_{n+1}||$ and $||H_n||$) that are not themselves the result of other quantitative operations. As illustrated in Figure 4 below, conceptualizing a difference includes reasoning about three quantities in relationship to each other.

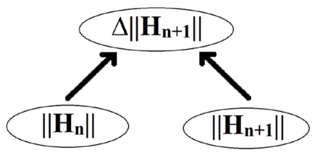
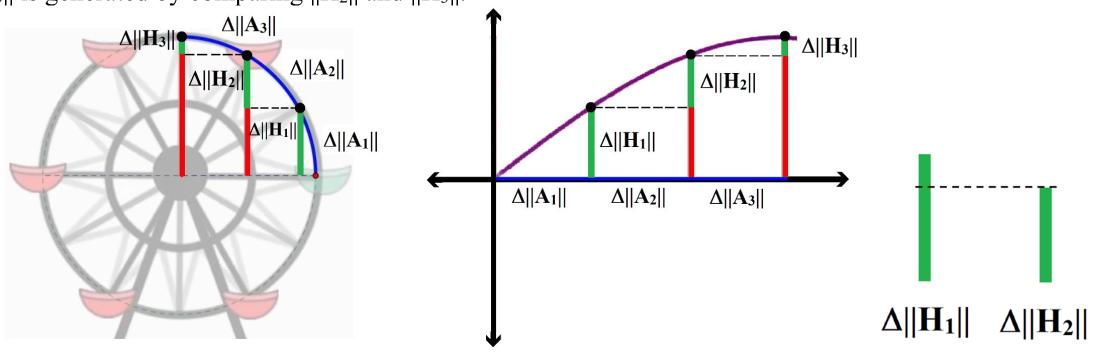


Figure 4. Quantitative relationship between three quantities.

Phase 2: Operations on amounts of change magnitudes

As seen in Figure 5a, recall from the basic case that $\Delta||H_2||$ is generated by comparing $||H_2||$ and $||H_1||$. Similarly, $\Delta||H_3||$ is generated by comparing $||H_2||$ and $||H_3||$.



(a) (b)

Figure 5. Operation on amounts of change magnitudes.

In this phase, the student's mental operation involves gross additive comparisons of amounts of change magnitudes (i.e., $\Delta||H_1||$, $\Delta||H_2||$, and $\Delta||H_3||$). Thus, the person forms a quantitative difference of differences (i.e., $\Delta(\Delta||H_1||, \Delta||H_2||)$ as seen in Figure 5b) and concludes the change has decreased from $\Delta||H_1||$ to $\Delta||H_2||$. This is a more complex quantitative reasoning because this requires relating groups of quantitative operations (i.e., additive comparison of two additive comparisons). This conception allows a student to conceptualize a difference of differences by coordinating the relationship among seven quantities, as shown in Figure 6.

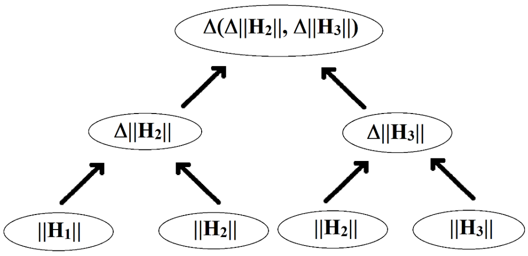


Figure 6. Difference of differences. Adapted from Smith III & Thompson (2008).

Phase 3: Patterns of amounts of change magnitudes

The Ferris wheel situation—similar to many other dynamic situations—involves quantities that vary in relation to each other (e.g., the height of the rider varies continuously in relation to the arc length). This feature of the situation (i.e., involving varying quantities) allows students to engage in repeated additive comparisons among $\Delta||H_1||$, $\Delta||H_2||$, and $\Delta||H_3||$. Therefore, students can conceive a pattern of differences (i.e., "identify a regularity" [Ellis, Tilemma, Lockwood, Moore, 2017]) and, in turn, generalize that $\Delta||H_n||$ decreases as n increases, as seen in Figure 7. Namely, a student can conclude $\Delta||H_1|| > \Delta||H_2|| > \Delta||H_3||$. Furthermore, the student might generalize that this pattern holds for any equal partition size in arc length within that quarter of a rotation. Therefore, in the context of Ferris wheel, this means that, by conceiving the pattern of differences, the person can describe the amounts of change in height with respect to uniform increments of change in arc length decreases as the rider travels in the first quarter of rotation.

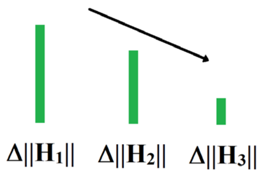
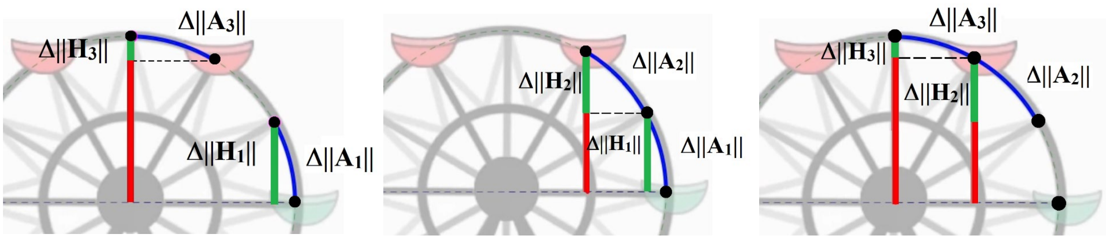


Figure 7. Pattern of amounts of change magnitudes.

Implications

Implications of use of phases in relation to students' generalization

We believe this conceptual analysis provides an additional lens for researchers to elaborate the process in which one might determine the invariant relationships between two covarying quantities in dynamic events. In particular, we believe the structure of the conceptual analysis including the three subsequent phases provides a structure for researches to discuss different implications depending on the nature of students' quantitative and algebraic reasoning in each phase.



(a) (b) (c)

Figure 8. Only two cases.

For example, with respect to Phase 1, researchers can describe different conceptions and, in turn, the implications of those differences in terms of students' potential generalizing actions. As an example, recall that, in Phase 1, someone considers operating on successive *cases* (i.e., considering different states of the height's magnitude one by one successively form the 3:00 position to 12:00 position as arc length's magnitude increases uniformly, see Figure 5a and Figure 7). Alternatively, someone may could only operate on two *cases* (see Figure 8). For example, the person could only create $\Delta||H_1||$ and $\Delta||H_3||$ in Phase 1, and then compare $\Delta||H_1||$ and $\Delta||H_3||$ conceiving $\Delta||H_3||$ is smaller than $\Delta||H_1||$ (see Figure 8a) in Phase 2. In Phase 3, without carrying out the operation of repeated additive comparison among quantities' magnitudes including $\Delta||H_2||$ (there could be more), the person could anticipate the pattern of differences. In this case, the person does not form a generalization from multiple cases across multiple iterations of $\Delta||H_n||$ (i.e., $\Delta||H_1|| > \Delta||H_2 > \Delta||H_3||$). Instead, the person predicts that the certain property (i.e., the decrease in amounts of change in height) will continue to other cases, even if this has not been observed. Although this generalization might hold, we note its limitation due to stemming from one case, rather than a collection of cases.

As another example, a different form of Phase 2 from that above involves a student who engaging in additional quantitative operations, which thus has different implications regarding the rest of phases. Recall that Phase 2 describes a student's operations on the magnitudes of amounts of change in height at successive states. An additional form of this phase can involve a student making multiplicative comparisons between $\Delta||H_n||$ and $\Delta||A_n||$ (instead of comparing $\Delta||H_n||$ where n varies). This form of reasoning is related to MA4 (Carlson et al., 2002), where the student constructs average rates of change of one quantity with respect to the other. Thus, this form of reasoning has implications for students constructing instantaneous rates of change.

Implications for graphing and reasoning about change

Researchers (e.g., Carlson, 1998; Carlson et al., 2002; Frank, 2017; Saldanha & Thompson, 1998; Moore & Silverman, 2015) have provided empirical evidence of how conceptualizing a quantitative structure allows students productively construct and interpret graphs. A student who engages with the conception that we illustrated here can determine how two quantities in a situation vary in relation to each other (e.g., increasing at decreasing amounts) and represent this structure in a different contexts or situations. It is important to note that when we say *a situation*, by following Moore (2016), we are not only referring to a situation of a real life context (i.e., "student's experiences with physical phenomena," p. 326). A displayed graph (or even tables and equations) can also be a situation where students can conceptualize a quantitative structure. No matter the type of number of representational systems, it is productive for students to engage in constructing quantitative relationships and reflecting on these relationships.

In relation to students' construction and interpretation of graphs

A student can engage in these phases in two ways: (i) in conceiving a situation (e.g., a dynamic real-world situation) as a quantitative structure, and reflecting on the quantitative structure when representing it in a coordinate system; or (ii) in constructing quantitative relationships that are represented in a given graph and making sense of that invariant covariational relationship as it relates to some situation. Those two different ways of engaging in these phases may characterize the difference between *constructing* and *interpreting* graphs (Ponte, 1984, p. 23). The former can be used in-the-moment of constructing graphs where the real-world context is the "situation" while the latter can be used to understand the information that the graph conveys about the situation and/or to justify the shape of an already constructed graph where the graph is the "situation." Engaging in such reasoning can aid students in avoiding translating perceptual features of a situation to a graph, or vice versa (e.g., the situation is a circle so the graph is a circle). Researchers have identified that such perceptual reasoning does not provide a foundation for productive and generative generalizations (Clement, 1989; Ellis, 2015; Monk, 1992; Moore & Thompson, 2015).

In relation to students' reasoning about change

Ponte (1984) pointed out that "graph interpretation includes the process involved in interpretation of variation and in interpretation of variation in variation" (p. 22), and he asserted that interpretation of variation in variation requires students to reason intuitively and constitutes most complex and most difficult aspect of graph interpretation for students. However, as we illustrated in the paper, students could make sense of the variation in variation by reasoning quantitatively and covariationally, in particular, by engaging in MA3 (i.e., amounts of change). Such reasoning only requires students to make gross additive comparisons between quantities' magnitudes, which "produces essential non-numerical inferences about quantities" (Smith III & Thompson, 2008, p. 25). Researchers (e.g., Davydov, as cited in Bass, 2015; Carraher & Schliemann, 2000; Smith III & Thompson, 2008) showed that the operation of additive comparison is something even primary-school age children can conceptualize and reason about, including differences in relation to their constituent quantities. Thus, we believe that this is a form of reasoning that is available to upper elementary and middle grades students. Yet, it is clear that such reasoning is not focused on currently in the U.S. and some international curricula (Moore, Stevens, Paoletti, Hobson, & Liang, 2019; Thompson & Carlson, 2017). This is problematic given that other studies show that such reasoning leads to more productive meanings for algebra, calculus, etc. (Thompson, Hatfield, Yoon, Joshua, & Byerley, 2017).

We acknowledge that we have investigated this model of mental actions in a limited number of contexts (e.g., Ferris Wheel situation), although more than one. We suggest future work should extend it to other case examples.

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