The Development of Situational-Misconceptions in Math Problem Solving

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Abstract: In this paper I use the *conceptual change* theoretical framework, in order to describe and evaluate the productivity of teacher's in-situ support to *Situational misconceptions*. I illustrate a case in which two students work together in order to solve a problem in math. The focal point is their teacher's intervention, in which she attempts to get her students out of a messy situation. This analysis helps to understand, in fine grain, the process – and consequences – of avoiding refutation of non-productive narratives, before adding new ones - in the context of math problem solving. In addition, we learn that in the process of solution, narratives might not always compete: ideas from two, or more, narratives could be forged to new *Situational misconceptions*.

Keywords: Conceptual-Change, Situational-Misconception, Math Problem-Solving, Competing Narratives

Introduction

Misconceptions could be best described as "*qualitative incorrect explanations that are expressed prior to, and even after, formal instruction"* (Chi, 2012). These incorrect explanations are outcomes of naïve theories that are developed and retained; theories that are many times consistent with the information that one has about the phenomena at hand, and thus, seem coherent to him (Vosniadou and Verschaffel, 2004). This notion of coherence will not disappear easily - causing meaningful change of some misconceptions is quite difficult (diSessa, 2006).

Studies about *misconceptions* could be allocated to three theoretical frameworks, (diSessa, 2006). One line of work, claims that there are sporadic concepts that appear in various domains of knowledge, mainly scientific in nature, such as Physics, Biology etc. These misconceptions should be mapped and treated specifically, in order to preform *conceptual change* (diSessa, 2006). The other two lines of work adopt Kuhn's *Paradigm change* theoretical framework. Accordingly, research about misconceptions in these lines of work, focus on the gap between misconceptions and the appropriate "scientific explanation". This gap is illustrated by Carey (1988) as a mechanism that restricts *conceptual change* due to incommensurability between competing mental models. This incommensurability restricts the building of the intersubjectivity that is needed in order to form productive discourse which builds common knowledge, but also help to refute non-productive ideas (Matusov, 1996). If we look at it the other way around, awareness of an incommensurability might trigger the need to come to shared understanding, through the process of collaborative meaning making (Stahl, 2009). This leaves us with the question: what are the traits of productive conceptual change? In particular, how could manifestations of incommensurability overcome?

Several solutions has been provided in the literature. The origins of most of these approaches is the elicitation of cognitive conflict (e.g von Glasersfeld 1995) between the current mental model and a new mental model. Chi (2012) for example, suggest that the solution is in a meta-strategic level: teaching students to be aware of competing narratives to approach the explanation of a phenomena, will lead to the realization of a new – maybe incommensurable – explanation. These explanations could be critically evaluated and compared; eventually, and hopefully, lead to the conceptual change. The same dialectical principle - meeting competing narratives – is applied with *refutation text.* A meta-analysis about *refutation text* by Tippett (2010), reveals that conceptual change could be achieved when the text (1) Starts with a description of the phenomena, (2) explains the misconception, (3) refutes the misconceived explanation, and finally (4) provides a new explanation that makes sense to the learner. In congruence, Asterhan & Schwarz (2007) show that engaging in argumentative discussion helps overcoming misconceptions; thus intelligible discussion about competing narratives/mental-models, could lead to productive refutation of the "wrong" and learning the "right". This same principle was described in the context of math, as Schwarz, Neuman and Biezuner (2000) show how argumentative scheme positively affects change of the use of decimal numeral systems.

Math problem solving and conceptual change

Math learning-and-instruction might benefit from adopting conceptual change models (Vosniadou and Verschaffel, 2004), since learning math often requires meeting competing narratives and choosing a productive one (Sfard, 2007). When a student holds a rigid-and-faulty *narrative* about the way a problem should be solved, we can address it as a *Situational Misconception*. In order to facilitate a shift in the narrative, a teacher might need to follow the core ideas of conceptual change: In terms of Posner, Strike, Hewsonand Gertzog (1982), she needs to make the students acknowledge that the current path does not lead them anywhere and should be refuted (dissatisfaction). In addition she should make sure that the new concept is intelligible, plausible and efficient in the eyes of her students, if she would like them to adopt this new concept.

A math *problem* is defined by Schoenfeld (1985) as a question for which the solution path is not known to the solver (in contrast with an *exercise*). Many studies in this field investigate students' attempts to solve problems on their own, in small groups and with computers. Most of these studies adopt a constructivist approach, with an ultimate goal of a guiding students in order to become active agents for their own learning (Zimmerman, 2008). Thus, a teacher has an important role in scaffolding such learning; she needs to serve as a facilitator of the learning, rather than mere instructor (Abdu & Mavrikis, 2016). She needs to be highly prepared to support a solution for a problem that sometimes has an open end, and usually has various paths to the solution (Abdu & Schwarz, 2012; Abdu and Mavrikis, 2016). In a way, she needs also to "unlearn" some of the materials: Experienced math-problem-solvers usually will make automatic moves in their solution (Arcavi & Isoda, 2007). These moves are done unconsciously, since they are a part of the teachers' "tool kit". *Unlearning* is quite a challenge for teachers, and as a result many teachers fail to "connect" with the difficulties their students experience (Ben-David, 2007).

In this paper I claim that it is important to adopt the conceptual change theoretical framework for the teaching of math problem solving. I use *Situational Misconception* in order to describe and understand a sequence of events in which teacher gives ill support to a group of two students.

The context of the episode

Background

I analyze the behaviors of two 8th grade students who solve a math problem -"the city", together. The problem was designed for three double lessons. Students were instructed to use Geogebra – A software that affords the creation of dynamic geometrical and algebraic objects in a Cartesian domain (Stahl, 2009). It is a part of a full-year course in computer-supported collaborative math problem solving, led by an experienced teacher. The teacher was guided to adopt a constructivist stance – acting as a moderator of learning, rather than direct instruction.

The problem

I bring the *City* problem (Based on Prusak, Hershkowitz & Schwarz, 2012), as it was presented in the beginning of the first meeting, by the teacher.

> "...the city council is going to build an energy center[…].This energy center needs to provide energy for the heating and cooling of seven public institutions of the city. All of the 7 public institutions pay good, equal, money […] for the building of that energy center […].We need to decide where to put the energy center in a way that will make as many as possible institutions, and people, happy. Note, when the length of the tubes between the energy-provider and the consumer increases, more energy is lost, and the energy efficiency gets smaller."

From a mathematical point of view, this problem asks if there a point that is equidistant to *any* given seven points?

Preparing to support the solution

This is not an easy problem for 8th graders. In congruence with Prusak, Hershkowitz and Schwarz, (2012), the teacher, along with the research team, broke the problem to three stages. *First stage:* there are only three institutions and the distances between them are equal – they form an equilateral triangle. In this case, the equidistant point is the *angles bisectors' /perpendiculars' /medians' intersection*. Note that in the case of an equilateral triangle it does not matter, since the three loci are at the same point. The students are familiar with these three concepts. *Second stage:* the distances between these three institutions are not equal, and they form a general triangle. The question in this stage is, what is the point that is equidistant to the three vertices of a general triangle? The former stage leads the solver to a conflict at this stage. Whereas in the former stage the solution is quite simple, it does not work for the case of a general triangle. Instead, the point that is equidistant to all vertices of a general triangle is the (1) *center of its circumscribed circle*, and, (2) *meeting point of the three* *perpendicular bisectors*. Further elaboration about the math behind these solutions will be brought below. The students are unfamiliar with these concepts. *Third stage:* Solve the problem for the case of seven points. An equidistant point exists only when these 7 points are located on one circle. The equidistant point will be the (1) *center of that circumscribed circle* and (2) *the meeting point of all the seven perpendicular bisectors*. Again, the students are unfamiliar with these concepts.

In accordance with these stages, the teacher first instructs the class to investigate a simpler case of the problem (first stage) - an equidistant point to three points in an equilateral triangle. Later on they will try to solve it for a case of a general triangle and last – try to generalize their conclusions for a case of seven points.

The teacher also prepares a supporting narrative [N1] for the solution. In order to understand her support, let's talk math for a little bit. The Narrative that the teacher tries to promote could be explained in figure 1. (1) We take the midpoint of segment FG. From this point, I, we create a perpendicular. (2) We locate point J somewhere on that perpendicular. (3) Points F and G are equidistant to J, since triangles ΔFJI and ΔGJI overlap. (4) Segments FJ and GJ are equal.

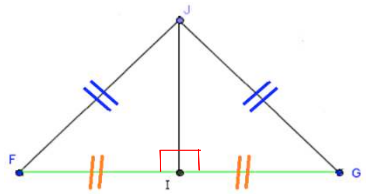

Figure 1. Teacher's hint - If we take the midpoint of segment FG and create a perpendicular, any point J on that perpendicular will be equidistant to F and G. Thus, FJ equals JG.

From here on we have two possible solutions. The first solution: Every given point, H (See figure 2), which maintains HJ=FJ=GJ, will help forming a triangle ΔFGH. The vertices of this triangle are all equidistant to point J. The conclusion in this case is: *The point (J) that is equidistant to all vertices of a general triangle is the center of its circumscribed circle*. In order to find such a point with the help of Geogebra, the students need to form a circle with Radius that is equal to GJ and FJ. The center of this circle will be located at point J. All the points that are on this circle are equidistant to point J, since it is the center of the circle; segments HJ, FJ and GJ are the radiuses of this circle. Therefore, any point H on this circle will be sufficient to create a triangle ΔFGH, with point J as the point that is equidistant from all of the vertices of this triangle.

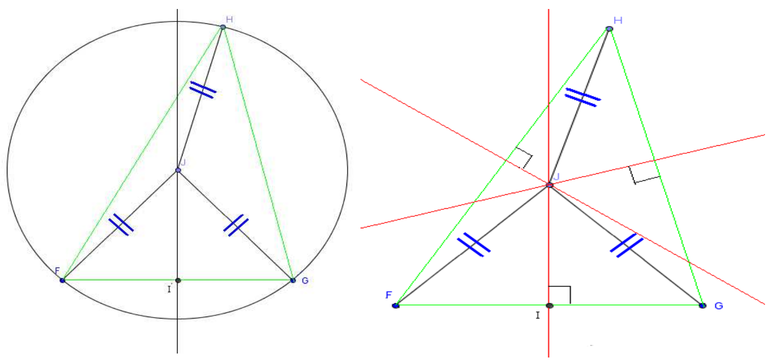

Figure 2. (Left) The point that is equidistant to all vertices of a general triangle is the center of its circumscribed circle. (Right) The three perpendicular bisectors of any triangle meet in one point; this point is equidistant from all the vertices of that triangle.

Another solution will be the three perpendicular bisectors of any triangle meet in one point; this point is equidistant from all the vertices of that triangle (same point as the center of a triangle's circumscribed circle). In order to find this point, let us remember that if we take the midpoint of segment FG and create a perpendicular, any point J on that perpendicular will be equidistant to F and G. Thus, FJ = JG (Figure 1). If we take section HG (figure 2, on the right), we can create perpendicular bisector that meets line GJ at point J. Since HJ=GJ and GJ=FJ, then FJ=HJ. Thus, point J is equidistant to all three vertices F, G and H.

Early stages of the solution

The team under scope has two students: *Halel* and *Amir*. They work together on two computers, while constantly monitoring each other's work. They pretty quickly come up with a shared understanding, like the rest of the class, that the equidistant point in the case of equilateral triangle is the angle-bisectors' intersection. They come to this understanding based on models they build with Geogebra. When Halel and Amir try to find a solution for a general triangle, they split. Each of them works next to his own computer, but they communicate orally at all times. Both come to the conclusion that the solution for the first stage is not valid in this case. They manage to find a solution for simpler case: The equidistant point from all vertices, in the case of *right-angled* triangle is the midpoint of the hypotenuse (see figure 3). On that note ends the *first lesson.*

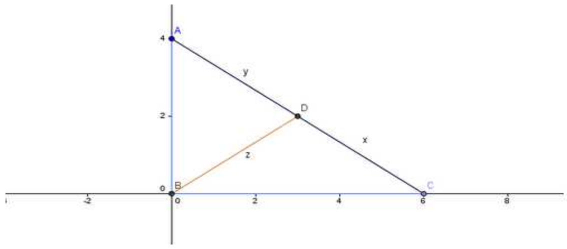

Figure 3. The point (D) is the midpoint of the hypotenuse, and thus is equidistant to all vertices of a right angled triangle.

The class goes back, for the *second lesson.* The teacher reminds the class what would be their upcoming steps - they need to find a point that is equidistant from the three vertices of any triangle. In addition, she gives a little hint that regards N1 - *"Your job now is to go to a general triangle, ok? ... Another thing that you can do, anyone that encounters difficulties, is to start with two points, not three, and then move on to three points"*,

Halel and Amir continue to solve the problem. They build a faulty narrative [N2]*: In the case of a general triangle, the equidistant point is on its longest edge.* They investigate this point and see that this hypothesis was wrong, and as a result they try make sense of their conclusions for the case of a right-angled triangle, trying to find out why it works for the right angled triangle but not for the case of a scalene. Amir raises another Narrative, [N3] *Solve the problem with the use of Pythagoras's theorem***.** This direction is irrelevant since the theorem is valid only for the case of a right angled triangles.

The episode: Teacher's intervention

The teacher joins Amir and Halel at this point, and after a quick attempt from the students' side to explain their hypotheses, the teacher recognizes that the directions they chose [N2 and N3] were non-productive. Since she wants the students to solve the problem together, she does not give them the solution and decides to assist them with the pre-planned support [N1], without *refutation* of their two narratives. She ignores their narratives and helps them to understand that there is more than one point that is equidistant from *two* points.

Teacher: [Looks at the Geogebra model in Amir's computer]: " *Is it possible to try…It is possible to try to get you out of this mess? ... Choose two points, anyone you'd like. Say, AB or whatever, does not matter just two points. Now, look for a point that is equidistant from A and B.* "

[Amir creates a median to AB, as could be seen in figure 4]

Teacher: "*Now, try to find another point that is equidistant from the two points.*"

Halel: "*Is it supposed to be on the median?*"

Teacher: "*What?"*

Halel: "*This point* [D] *is already equidistant...So everything that will be related to that will be equal!* [Screams] *Ahh, I got it!!! Move away!* [To Amir]"

Amir: [Looks at Halel] " *This is what I am trying to say to you for half an hour now!!! But you would not listen*".

[Amir and Halel turn, each one to his own computer]

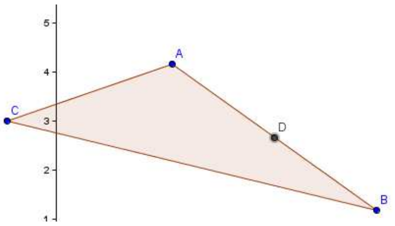

Figure 4. In triangle ABC, point D is equidistant from A and B.

At this point, three narratives are at stake. The first is the conjecture that the desired point is on the longest segment [N2], the second is that "it has to do with Pythagoras's theorem" [N3] and then there is the teacher's new narrative [N1].

Solution attempt 1

Amir: " We know that the triangle is right angled, and we know that […] this, square [Points at section AC] plus this, square [Points with his fingers to section CB] equals to this, square [Points at section AB]. So we can define them [AC and CB] as half [of AB]. Half of this [Points at section AB] equals to this half [Points with his fingers to section AC] OK?...So what I am saying is [that] when you take here the median [Points at section AB] …then it [AB] equals to the two halves [AC and CB]"

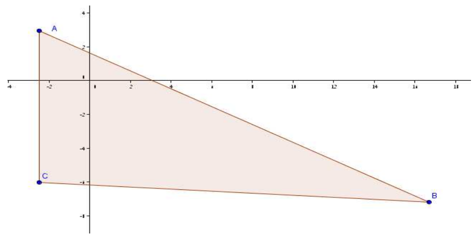

Figure 5. Amir's Geogebra model of a right angled triangle.

Amir explains now to Halel and the teacher why he thinks the solution for the right angled triangle is the middle of the hypotenuse. By doing so he will make two mistakes: First, he assumes that the formula $AC^2+BC^2=AB^2$ is equivalent to the formula AC+BC=AB (See Ben-David, 2007, about this particular misconception). Second, he assumes that if AC+BC=AB (which is completely wrong), then AC=AB/2. He starts with a hasty sketch of a right angled triangle. Amir uses these wrong assumptions and moves forward, based on these assumptions he assumes that these narratives will be applicable for the case of any triangle.

Amir: *"Then I moved from here to here* [Moves point 'A' so the triangle looks like one in the figure 5]... *I now try to play with this point* [Points with his fingers to segment AB] *which is exactly what I wanted to do* [Grabs the computer mouse, creates a midpoint D to section AB]. *So the midpoint* [of section AB] *…is a fixed point. Now I need to make another point* [adds a point to the drawing] *I put it on point D and can find [equidistant point]…*[Creates a point G]

Amir still thinks that the "point that is equidistant from all three vertices of any triangle, is on segment AB, which is the longest segment. His plan is to use the computer simulation, create a point [G] on segment AB and move this point along this segment in order to find the equidistant point [N2]. He bases his claim on mathematical manipulations that are derived from [N3]. In order to meet with the teacher's instructions [N1: all the points that are equidistant from A and B are on a perpendicular bisector], he integrates previous conclusion – in the case of a right-angle triangle, the point that is equidistant to all vertices bisects the hypotenuse.

Now, the teacher tells the students that she needs to leave, and without diagnosing how her hint was perceived, she takes off. After she does so, the two students keep hopping between solutions but do not appear like they are going to solve the problem.

Solution attempt 2

Amir builds a perpendicular bisector to AB, from its Midpoint, D. (see figure 6).

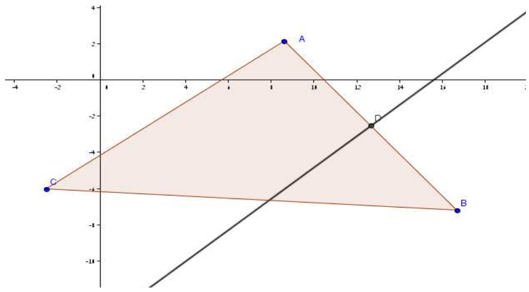

Figure 6. Amir's model - point D is in the middle of segment AB, and build a line that is perpendicular to this segment.

Amir tries to apply the teacher's narrative, for their case by building a perpendicular bisector. But he does not have the opportunity to develop narrative [N1] and Halel stops him with an attempt to build upon [N2] from another perspective.

Halel: "*Amir, can I do something?"*

Amir: "*Just a second*. [*Puts a general point on the perpendicular, and erases it.*] *OK, you go on*"

[Halel sits next to Amir’s computer, as Amir looks at what he does. Halel erases the perpendicular from point D, and creates a midpoint E to section CA, and a midpoint F to section BC (Figure 7). He then tries to create three segments – EF, FD and DE and erases them at once.]

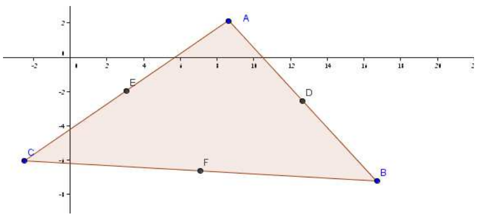

Figure 7. Points D,E,F are the medians of sections AB,AC and BC correspondingly.

The attempt to create a perpendicular bisector by Amir – which is closely related to [N1], is stopped by Halel. This might be a sign for Halel's inability to integrate this narrative into his mental model. Halel takes a step back and connects the three midpoint of the three segments. We cannot tell what Halel's ideas was, but it appears to be an integration of [N2] that claims that the point that is equidistant needs to be on one (longest) of sides of the triangle and [N1] that has to do with the creation of some sort of a segment that starts at the midpoints of the segments.

Let us note, in addition, that a productive path to the solution would have been a merge of the narratives of Amir and Halel: Creating three *perpendicular* bisectors from the three sides of the triangle will yield the point that is equidistant to all three vertices.

Discussion

The teacher's role in this study is of a facilitator that scaffolds the learning, rather than a direct instructor. As a result she gives the students hints, and not the solution. She comes with a pre-packed narrative, and disregards the narratives that were brought up by the students. Webb, (2009) explains that proper scaffolding of students' collaborative learning, requires an accurate and detailed analysis of the students' thinking before giving a prompt that is a best "fit" to those states. It is easy, but somewhat unfair, to critique a teacher's work, and I would hereby mention that the teacher involved is highly competent teacher. She wants her students to self-regulate their learning (Zimmerman, 2008). She also needs to cope with time limitation, in addition to the fact that she needs to move on and support other team's solutions. She has little time to diagnose the students' Situational-misconceptions, refute them, and to make sure that her hint was to lead the students anywhere. In addition, by all means I do not wish to claim that no learning was done. On the contrary, the technological scaffold, Geogebra, serves here as a tool that allows the students to refute their erroneous narratives (Hadas, Hershkoitz and Schwarz, 2006).

However, our obligation here is to use the conceptual-change framework in order to explain, where did she go wrong? In order to use the *conceptual-change* theoretical framework, in the context of online monitoring and regulation of collaborative learning, I define the term *Situational misconception* as a faulty narrative about the way the problem should be solved, which confines student's ability to solve the problem. Amir and Halel exemplify two main narratives [N2 & N3] that are situational-misconceptions. These situational-misconceptions are manifested as Geogebra models of possible solutions to the problem. The teacher instructs them to create a Geogebra model that serve as a manifestation of a third narrative. This is a technological scaffold (Pea, 2004) that was created by the teacher in order to foster the solution. As a result of their interaction with their teacher, they were introduced to an additional narrative [N1] that led to a couple of solution attempts that *failed*, since this narrative [N1] was incommensurable with their previous narratives [N2 & N3].

Failure of solution attempt 1

This solution attempt starts with [N3], which relies on the faulty assumption that there is a relation between the fact that the equidistant point is in the midpoint of the hypotenuse and Pythagoras's theorem. The *midpoint* appears in the teacher's hint [N1] as well, but has a completely different role: It defines the point from which a perpendicular is built in order to find all the points that are equidistant to the two vertices of the triangle's segment. Consequently, this does not change the students' assumption that the equidistant point in the case of scalene triangle is somewhere on the long edge [N2]. This leads also to the failure in solution attempt 2.

Failure of solution attempt 2

The equidistant point in the case of a right angled triangle appears on one of the sections (the hypotenuse). Thus, the equidistant point in the general case might be on one of the sections (wrong!) [N2]. Since the teacher's hint involved a point that is on one of the vertices – a midpoint [N1] – then if we build midpoints for the three segments of the triangle, and connect them, we might find a point that is equidistant from the three vertices.

Sfard (2007) uses Kuhn's term "incommensurability" in order to describe the use of the same term in two different conceptual frameworks. Sfard suggests that the interlocutors (students and teacher) should explicate how does the term they are using fits in their narrative, in order to proceed further. In our case, this is not a term, but a mathematical concept that was applicable in both narratives. In our case, this concept was the *midpoint.*

Posner, Strike, Hewsonand Gertzog (1982), claim that the teacher should make the students acknowledge that the current narrative does not lead them anywhere, before she adds a competing *plausible* narrative. Eventually, the added narrative [N1] confuses them, as they try to assimilate it to their narratives, instead of applying deeper change (accommodation). Nonproductive narratives were not refuted, which yielded the creation of two solution attempts, based on the use of the same concept – in different contexts. Accordingly, these two solution attempts are hybrids of all three old narratives [N1, N2 & N3].

Conclusions

From a theoretical perspective, the current work makes an attempt to analyze teacher's intervention in math problem solving, from the lens of the *conceptual change* framework. I use the term *Situational-misconception* to describe a nonproductive narrative; and see how the addition of a narrative to that situational-misconception, without refutation, leads to farther situational misconceptions. The adoption of such a framework serves fine-grained apprehension of how a misconception develops. It allows us to understand *why* the literature on *Conceptual change* adheres refutation of non-productive narratives, before adding new ones.

Moreover, we learn that the lack of refutation does not necessarily lead to continuity of the "old" misconception. While the literature on conceptual change mainly attempts to describe successful cases in which faulty concepts are replaced by new ones, the novelty of the *situational misconception* framework regards to the identification of a twilight-zone: two competing narratives do not really impede each other, and ideas from both new and old narratives merge into new misconceptions.

This study contributes to in-class practice, by re-encouraging teachers to listen to their students' narratives carefully and refute them in a plausible way, when needed, before they add any competing narrative. In addition, teachers might find it beneficial to be highly familiar with possible misconceptions on the path to the solution of a problem in math. To rest our minds, it is important to mention that later on the teacher intervened in Halel and Amir's solution again. This time she was much more explicit; and still did not refute her student's nonproductive narratives. This led them to solve the problem.

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