

Supporting Calculus Learning Through “Smooth” Covariation

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Abstract: Calculus is a passport for STEM-related careers. In response, this thought experiment analyzes the current nature and state of pre-calculus, drawing attention to a common and problematic approach to teaching function. I argue that this approach to function fosters a kind of thinking that conflicts with the kind of thinking required for a conceptual understanding of calculus and suggest one way to support teaching and learning of pre-calculus is to highlight the features of function that are fundamental to calculus. I suggest that this can be done through carefully constructed mathematical tasks and present a framework for task development.

Keywords: covariation, pre-calculus, calculus, smooth function

Introduction

According to the Bureau of Labor Statistics in May 2014, the ten highest annual mean salaries by occupation are in STEM-related fields. STEM-related fields are the most lucrative and fastest growing fields in the United States and many other countries. Candidates who are eligible for STEM-related careers usually have a strong mathematics background. Unfortunately, for many students, calculus serves as a “gatekeeper” on the education path that qualifies students for a STEM-related career. In turn, students are marginalized from the career and economic opportunities afforded by STEM-related careers.

As the global economy continues to advance, education must evolve and advance with it. In the United States, there are not enough students taking pre-calculus courses in high school. In 2009, 35% of high school graduates took pre-calculus (National Center for Education Statistics, 2015). Given the economic affordances of calculus, these numbers are alarming. I argue that the problem of calculus stems from the nature of teaching and learning calculus.

Similar conclusions have been drawn about algebra. In response, researchers have argued for *early algebra*, which they distinguish from algebra early. Early algebra is a way of thinking that spans elementary and middle school mathematics. It involves building on traditional elementary curriculum by generalizing, representing, justifying, and reasoning with mathematical structure and relationships (e.g., Blanton, Levi, Crites, & Dougherty, 2011; Carpenter, Franke, & Levi, 2003; Kaput, 2008; Kaput, Carraher & Blanton, 2008). I argue that the first step to addressing the problem of calculus is to re-conceptualize pre-calculus as a longitudinal strand of thinking that extends through middle and high school mathematics. That is, pre-calculus should involve building on algebra by recognizing and articulating change and continuity in functional relationships. This is a new and evolving idea, but I suggest one way to support this approach to teaching and learning pre-calculus is to highlight the features of function that are fundamental to calculus, and I argue that this entails characterizing functions as “smooth,” continuous, covarying quantities.

Despite the many algebra and pre-calculus curricula that use function as a central concept, most high school and freshman college students have a weak understanding of function, which is an important concept for studying advanced mathematics (Carlson, 1998; Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Monk & Nemirovsky, 1994; Thompson, 1994a). I argue that the way that function is approached in teaching is conveyed through curriculum and instruction, including problems and activities. Moreover, the instructional approach fosters conceptualizing function a certain way.

Often problems about or approaches to teaching functions promote thinking about variation at whole number intervals. For instance, a problem that reinforces this approach might ask students to graph a given function. To solve this problem, students will likely plot and connect points. While students may correctly graph the function, it is unlikely that they are making sense of how one quantity changes as the other quantity changes or attending to the change that occurs between the whole number intervals. In other words, students are unlikely to interpret the function in terms of two continuous covarying quantities because a task similar to the one described draws attention to specific x - and y -values, in turn, portraying function as discrete and variables as static. This approach is commonly used by students in U.S. schools (Leinhardt, Zaslavsky, & Stein, 1990).

Despite its prevalence, researchers believe this approach over emphasizes procedural and localized skills (Bell & Janvier, 1981; Leinhardt, Zaslavsky, & Stein, 1990). Furthermore, I argue it overlooks the characteristics of function that are essential to finding a derivative because it avoids the continuous features of function, and furthermore highlights the discrete features of function. That is, this approach steers away from a view of function that supports learning in calculus, and fosters a kind of thinking that conflicts with the kind of thinking required

for a conceptual understanding of calculus. In response, this thought experiment explores a covariational approach to function that promotes learning in calculus and suggests characteristics of mathematical tasks that support the approach. Specifically, I propose a framework for constructing mathematical tasks based on two arguments. First, “smooth” covariational reasoning fosters a productive disposition towards calculus. Second, the mental action of coordinating covarying, nonnumeric quantities promotes “smooth” covariational reasoning. In the following section, I justify these arguments. Then, I present the proposed framework. Finally, I share the theoretical and practical implications of these ideas.

Fostering a productive disposition towards calculus

Covariational reasoning is the mental action of “coordinating two varying quantities while attending to the ways in which they change in relation to each other” (Carlson et al., 2002, p. 354). A covariational approach to function is critical to students’ understanding of numerous secondary mathematics topics including quadratic relationships (Ellis, 2011), exponential relationships (Castillo-Garsow, 2012; Confrey & Smith, 1995), trigonometric relationships (Moore, 2010; Moore & Carlson, 2012), rate of change (Carlson et al., 2002; Thompson, 1994a), the conceptual ideas of calculus (Thompson, 1994b), and differential equations (Rasmussen, 2001).

By definition, reasoning covariationally necessitates reasoning about covarying quantities in terms of rate of change. Hence, prior research has shown that reasoning covariationally results in thinking about change in a sophisticated way that is constructive to understanding calculus (Johnson, 2012). Moreover, the covariational approach to function draws attention to specific types of change (e.g. difference, accumulation and rotation), and distinguishing types of change is critical for expressing and describing a relationship between two quantities in multiple ways (Confrey & Smith, 1995). By focusing on change, a covariational approach steers away from the discrete features of function, and furthermore highlights the continuous features of function, in turn, fostering the kind of thinking required for a conceptual understanding of calculus. Namely, the covariational approach exposes the “conceptual underpinnings” of calculus (Johnson, 2012, p. 313).

If students are to be calculus-ready, they need experience with covariational reasoning about function. In particular, types of covariational reasoning align more closely with the concepts of calculus. For instance, researchers have distinguished “chunky” forms of covariation from “smooth” forms of covariation (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013). “Chunky” reasoning involves thinking about change in complete chunks (Castillo-Garsow et al., 2013). In contrast, “smooth” reasoning involves thinking about change as an ongoing process. A “chunky” approach is fundamentally discrete because it is built on integers, whereas, a “smooth” approach is fundamentally continuous, because it is built on the real numbers. Research in the area of “smooth” and “chunky” reasoning is emerging. However, researchers agree that “smooth” thinking is critical when exploring the ideas of calculus (Castillo-Garsow, 2012; Castillo-Garsow et al., 2013; Johnson, 2012). Thus, “smooth” covariational reasoning fosters a productive disposition towards calculus.

Promoting “smooth” reasoning through nonnumeric quantitative reasoning

Mentally operating on quantities to construct new quantities, while attending to the relationship between the quantities that were operated upon to construct the new quantity, is quantitative reasoning (Thompson, 1994b). Thus, covariational reasoning is reasoning quantitatively with covarying quantities. Furthermore, Thompson (1994b) argues that at its core quantitative reasoning, fundamental to covariational reasoning, is nonnumeric. That is, quantities are not necessarily numbers; numbers are specific quantities. The process of assigning numerical values to quantities is quantification, but quantification is not a requisite for conceiving a quantity (Thompson, 1994b). Hence, no numeric measurement is necessary to reason with quantities, and quantitative reasoning can be nonnumeric (Johnson, 2012; Thompson, 1994b). I argue that nonnumeric quantitative reasoning can support “smooth” covariational reasoning.

One conceives distance as a varying quantity, when she reasons about a runner’s location as a measure of distance, d , from the start of the run. Furthermore, one is reasoning covariationally and quantitatively when she conceives time, t , as a quantity that simultaneously varies with d (Thompson & Carlson, in press). That is, one reasons covariationally about the runner’s speed when she conceives rate of change as the ratio of two quantities, d and t . However, if one has a static conception of variable—variable is a symbol for unknown values—then, constructing that ratio does not result in “smooth” covariational reasoning (Thompson & Carlson, in press). In other words, one can conceptualize speed as a ratio without conceiving d and t as continuously changing, but rather, by envisioning constant values for d and t and then comparing those values to conceive the ratio. By doing this one does not attend to the change between the constant values, thereby the reasoning is covariational and “chunky.” This aligns with Confrey and Smith’s (1995) definition of covariation.

An example of a “chunky” description of speed is “for every eight minutes, distance increases by one mile.” This ratio could result in “chunky” reasoning because the relationship is built on positive integers (Castillo-

Garson, 2012). Without specifying integer value, quantities have no constant value, and then one must relate quantities, thereby attending to change in the quantities, to operate on them, or to reason quantitatively. Thus, I suggest that representing quantities without numeric measurement promotes “smooth” covariational reasoning. For example, reasoning about a runner’s distance in relation to the start of her run is reasoning about a quantity with a magnitude, but no numeric measurement. I suggest that nonnumeric quantitative reasoning promotes “smooth” covariational reasoning because it draws attention to the continuity of the variation.

Framework for constructing tasks

In this section, I present a framework for constructing tasks that might support “smooth” covariation. The first three criteria address the context of the task. The last three criteria address the mathematics of the task and will be elaborated on in what follows.

First, the task should involve function. Second, the function should be presented as nonnumeric. Third, when possible, the function should model an experiential situation. For instance, riding in a car is experiential, whereas the growth of a bacteria colony is not experiential. Fourth, tasks should avoid the action view of function (Dubinsky & Harel, 1992). Fifth, tasks should allow for association (Thompson, 1996; Johnson, 2012). Lastly, tasks should involve monotonic functions (Ellis, personal communication, November 3, 2015).

The action view of function involves representing a function as an algebraic expression, whereby students can plug values in for the independent and dependent variables (Dubinsky & Harel, 1992). Such an approach does not prevent students from “smooth” reasoning. However, it is conceptually limiting. Specifically, Dubinsky and Harel (1992) contrast the action view of function with the process view of function and argue that the process view “involves a dynamic transformation of quantities.” Essentially, the process view allows students to “think about the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done” (p. 85). Hence, the process view of function aligns with quantitative reasoning. Therefore, I suggest that tasks avoid the action view in order to foster quantitative reasoning about function.

Association refers to the mental connections that students construct between objects and attributes of objects (Thompson, 1996). For instance, when considering an area problem, a student is using association by relating side length, perimeter, and area to reason mathematically (Johnson, 2012). Association enables acting on objects, and acting on objects facilitates internalization of the objects and the actions. In turn, association promotes constructing a mathematical mental image. Thompson (1996) claims that images contribute “to the building of understanding and comprehension” (p. 16) and support reasoning about quantitative relationships (p. 15). Therefore, tasks that offer space for association support mathematical image construction, thereby promoting understanding and quantitative reasoning.

A monotonic function is a function that is either always non-increasing or always non-decreasing. Namely, the rate of change does not switch signs. Through personal communication with Amy Ellis (November 3, 2015), I realized the relationship between a function’s monotonicity and “smooth” reasoning. Reasoning about a monotonic function supports “smooth” reasoning because there are no points at which the direction of the variation changes.

Conclusions and implications

As the global economy continues to advance technologically, STEM-related careers become more competitive and lucrative. If students are not prepared from an early age to take calculus they are marginalized from the career and economic opportunities afforded by STEM-related careers. At a practical level, I address the lack of calculus preparation in current mathematics curricula. Furthermore, I argue that the current approach is problematic, and potentially detrimental to students’ success in calculus. On a theoretical level, I contribute a framework for constructing tasks that promote students’ development of a productive disposition towards calculus. Future research could build on this work by scrutinizing and refining the framework. Moreover, a design experiment that iteratively tests the framework in the classroom could inform instruction and curriculum development. However, I recognize that tasks, which fit these criteria, do not guarantee “smooth” covariational reasoning. I also acknowledge that there are many ways to support calculus learning, and that fostering “smooth” covariational reasoning is only *one* approach.

This paper is a thought experiment, and I welcome criticism. I invite readers to share challenging perspectives and tasks that fit the framework criteria.

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