The Construction, Refinement, and Early Validation of the Equipartitioning Learning Trajectory

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Abstract. Equipartitioning, a foundational construct of rational number, is introduced in the form of a learning trajectory. Its development is linked to synthesis of research, the conduct of clinical interviews, and the development of related diagnostic assessments. Refinement and validation of a learning trajectory is described as an iterative empirical process that can serve as a prototype for design of other learning trajectories.

There is perhaps no more important conceptual area in mathematics education than rational number reasoning. The basis of the multiplicative conceptual field (Vergnaud 1983, 1996), rational number reasoning underpins algebra, higher mathematical reasoning, and the quantitative competence required in science. Failure to develop robust rational number construct reasoning and skills in elementary and middle school continues to plague American students. Rational number reasoning is complex, and mastery represents cognitive synthesis—understanding, distinguishing among, modeling, and interweaving a remarkable assortment of distinct yet closely related concepts over many years.

More than four decades of research in rational number topics have provided us with a robust outline of the concepts that comprise rational number reasoning. As described over years of research in the Rational Number Project, and recently outlined by Confrey and colleagues (Confrey 2008; Confrey et al. 2009), rational number reasoning encompasses the constructs of equipartitioning, multiplication and division, ratio, rate and proportion, scaling, length and area measurement, fractions, decimals and percents. That research and summary of rational number strongly suggests that while Rational Number Reasoning is complex, it will yield to a Learning Trajectories analysis (Confrey et al. 2009).

Once learning trajectories are developed, it is assumed that they can be deployed as a framework for curricular innovations and for strengthening teachers' understanding of and instruction in rational number. When coupled with innovative cognitive diagnostic psychometric approaches, learning trajectories may provide a robust means for teachers to assess students' cognitive progress.

"Learning trajectory" is a term with roots in Simon's (1995) definition of "hypothetical learning trajectory," a teacher-conjectured path of student learning from beginning to end of an instructional episode. Learning trajectories are also described in the research literature as predictable sequences of constructs that capture how knowledge progresses from novice to more sophisticated levels of understanding. Clements and Sarama (2004) suggested somewhat more predictable, larger-scale structures, "descriptions of children's thinking as they learn to achieve specific goals in a mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (p. 83). Science educators define *learning progressions* as "empirically-grounded and testable hypotheses about how students' understanding of, and ability to use, core scientific concepts and explanations and related scientific practices grow and become more sophisticated over time, with appropriate instruction" (Corcoran et al. 2009, p. 8). Other views of learning trajectories include the process of developing key conceptual structures (Catley et al. 2004) and that instruction should trace a prospective *developmental* (Brown & Campione 1996) or *conceptual*, *corridor* (Confrey 2006; Lehrer & Schauble 2006), spanning grades and ages, with central concepts introduced early in the school experience and progressively refined, elaborated, and extended. We use a refined definition of "learning trajectory" (or "learning progression"):

> Learning Trajectory: A researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey 2008; Confrey, Maloney, Nguyen et al. 2008; Confrey et al. 2009).

While learning trajectories do not dictate a hard-and-fast order in which topics must be learned for students to be successful, they permit specification, at an appropriate and actionable level of detail, of ideas students need to know during the development and evolution of a given concept over time. This information can be embedded systematically in state curricular standards (Confrey & Maloney in press), can underlie curricular interventions with varying degrees of explicitness, and guide teachers in responding to and anticipating student thinking. Learning trajectories may form the backbone of an assessment strategy that can both document

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student progress and identify individual and subgroup weaknesses that can be specifically and efficiently addressed by teachers.

The DELTA project (Diagnostic E-Learning Trajectories Approach to Rational Number Reasoning) at North Carolina State University has the multiple objectives of creating learning trajectories for up to seven distinct strands of rational number constructs, validating them through empirical study, and then designing diagnostic assessments, to assist teachers in targeting their instruction to more effectively support students' cognitive progression through the key rational number constructs across grades K-8.

Methodology for construction, refinement, and early validation of Equipartitioning learning trajectory.

To construct a learning trajectory, an exhaustive collection, review, and synthesis is conducted of mathematics education and cognitive psychology research literature pertaining to topics in rational number. Synthesis demands careful attention to making sense of related studies, explain higher order relations in complex cumulative findings, or explain old findings by postulating new concepts (Cooper 1998). Once the initial learning trajectory is identified, additional open-ended semi-structured clinical interviews (Piaget 1976; Opper 1977) are conducted. In the case of equipartitioning, these interviews were conducted with 42 urban and suburban students in grades pre-K to 6, and lasted 30 to 60 minutes each. Interviews presented participants with various sequences of specific tasks (varying the numbers of items and people in the fair sharing activities) and followed up with particular questions designed to elicit children's explanation of strategies they used to accomplish the tasks, their justification for their solutions to the tasks, and to probe for multiple strategies for accomplishing each particular version of each equipartitioning task. The interviews permit filling in gaps in the learning trajectory.

The next step in the process[1] is to develop paper-and-pencil assessment items, which requires further clarification of the levels of the trajectory, and piloting of the items. Simultaneously, think-alouds are conducted to determine the extent to which student responses on paper express the range of student thinking elicited by the item. In the DELTA research, 14 items were piloted with 95 2nd-grade students, and 14 think-aloud interviews were conducted (Wilson 2009). Based on the synthesis work and the interviews, outcome spaces--the range of observed cognitive behaviors--are summarized for each level of the learning trajectory.

Field-testing follows the development of items corresponding to each level of the learning trajectory. In the equipartitioning research, 125 items were created and administered to approximately 5000 students, grades K-7, across 4 North Carolina school districts. All the students in these grades in 7 elementary and 3 middle schools completed the field tests.

Item-scoring rubrics are then developed in a two-stage process: a) small teams examine all responses to an item, draft a rubric, and identify a range of student response exemplars; b) the team presents the rubric and exemplars to the entire research team, at which time the correspondence of the responses to the learning trajectory levels is evaluated. For the equipartitioning LT, rubrics comprised a 3- to 6-level scale, depending on item format, that accommodated the range of sophistication of item responses, from non-response (or unintelligible) to unequivocally correct or complete. Responses lacking sufficient evidence to be categorized were also coded. Other codes were created to facilitate collection of data on particular strategies and misconceptions.

Defining Equipartitioning. Using cases to build an initial learning trajectory.

A broad variety of mathematics education and cognitive psychology literature bearing on as many readily identifiable aspects of rational number reasoning as possible was assembled into a comprehensive database (approximately 650 publications). The intent was to assemble an exhaustive collection of peer-reviewed journals, edited books and chapters, and conference proceedings (http://gismo.fi.ncsu.edu/database) (Confrey 2008; Confrey, Maloney, & Nguyen 2008). In examining this diverse database for studies of children's early understanding of multiplication and division, fraction, and ratio, to identify evidence of common or earliest cognitive roots of the concepts in rational number reasoning, it was noted that investigators across a diverse range of differently-oriented studies had asked participants in their studies to solve fair sharing activities. These authors had investigated a variety of mathematical and cognitive topics, including one-to-one correspondence (Frydman & Bryant 1988), partitioning (Pothier & Sawada 1983), early fraction understanding (ref), unitizing (Lamon 1996), the partitive fraction quotient construct (Charles & Nason 2000) (Toluk & Middleton 2003), early ratio understanding (Streefland 1991; Confrey & Scarano 1995; Confrey & Lachance 2000), and aspects of fair sharing itself (Hunting & Davis 1991), and the cognitive relationship between fair sharing and counting competency in young children (Pepper 1991; Pepper & Hunting 1998). Many of the authors had observed that children's reasoning in fair sharing activities followed predictable pathways of increasing sophistication. Early on, Pothier and Sawada (1983) explicitly recognized connections between partitioning and rational number ideas, and identified number theoretic properties and properties of particular simple geometric shapes as important components of children's developing reasoning. Students performing

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these fair sharing activities often gave evidence of reasoning about phenomena predicted by the splitting conjecture (Confrey 1988; Confrey & Scarano 1995).

These fair sharing activities possessed the common feature of generating equal-sized groups or parts, an objective initially named by the research team as "partitioning." The partitioning in these cases, however, is distinct from the kind of partitioning identified in part-part-whole number sense. *Equipartitioning,* as it has been named is partitioning with the specific objective of generating equal-sized groups. We explicitly distinguish the term "equipartitioning" from "breaking," "fracturing" or "segmenting" in which the goal is not necessarily the creation of equal-sized groups, and from the general mathematical term "partitioning." The definition of equipartitioning is as follows:

> Equipartitioning: cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing "fair shares" for each of a set of individuals (Confrey et al. 2009).

Four distinct cases of fair sharing activities were identified as a means to unify equipartitioning (Confrey, Maloney, Nguyen et al. 2008). These four cases comprised the original framework of the Equipartitioning learning trajectory (EqPart LT). The cases are listed here, each with an example of one of the familiar contexts:

- Case A: sharing a collection of m objects fairly among p people, where p is a factor of m. Example: 15 coins of treasure shared fairly among 3 pirates;
- Case B: sharing a single (dissectible) whole fairly among p people. Example: 1 rectangular birthday cake shared fairly among 4 people.
- Case C: sharing multiple continuous wholes (m) fairly among p people, where $m < p$, and producing m/p as a proper fraction. Example: 3 circular cakes shared fairly among 4 people.
- Case D: sharing multiple continuous wholes (m) fairly among p people, where $m > p$, and producing m/p as an improper fraction or a mixed number. Example: 9 pizzas shared fairly among 4 people.

The initial learning trajectory was essentially a simple-to-complex ordering of the cases A-B-C-D, and accounted for a number of previous findings. For instance, studies of one-to-one correspondence had established that dealing was a fundamental strategy for fair sharing of collections. Studies of partitioning and splitting had already shown that 2-splits of collections or of single continuous whole objects were readily accomplished by very young (3-4 years old) pre-literate children (Hunting & Sharpley 1991). It had also been established that students mastered different levels of splits in a distinct order, but that the order of mastery did not follow a counting sequence: students consistently accomplish 2- then 4- and commonly 8-splits before accomplishing other split-values, and that 3-splits are very difficult, especially on circular shapes. It was also clear that non-even splits of circles presented far more difficulty for students than splits based on powers of 2 (Pothier & Sawada 1983; Confrey 1997).

Iterative development of the Equipartitioning learning trajectory.

Articulating and refining the EqPart LT is an iterative process of research synthesis and empirical investigation involving clinical and think-aloud interviews, assessment item response analysis, and rubric development. Previous studies used activities of fair sharing collections of discrete items or single wholes with younger children (pre-K through grade 3), and the more complex tasks of fair sharing multiple continuous items with older elementary and middle school-aged children. However, no systematic examination of the progression of children's reasoning across a wide range of ages and across the entire set of four equipartitioning cases had been conducted. To systematically investigate children's mathematical reasoning in the context of fair sharing across all four cases, semi-structured clinical interviews were conducted with children grades pre-K through 6.

The clinical interviews were designed to elicit a range of student reasoning, including strategies, representations, mathematical practices, emergent properties, and misconceptions. Each interview probed a variety of task parameters, i.e. evaluating students' ability to accomplish tasks of fairly sharing various numbers of items for various numbers of people, and to describe the results of reversing the activity, i.e. reassembling the shares. For each combination of items and people among whom the item(s) were to be shared, students were also asked to verify that each person had received a fair share, justify how s/he knew this, to name the fair share, to identify how much of the collection or item(s) each person received, to share the collection or item(s) in at least one other way, and to identify one n^{th} of the total collection or item(s). Additional questions, designed to elicit student reasoning about the consequences of, for instance, changing the numbers of persons or items at the conclusion of one task, were also employed. A typical interview task, for instance Case A, involves providing a child with a set of plastic "gold" coins, telling her that two pirates have discovered a treasure, and they want to share the treasure fairly. The interviewer then proceeds to ask the child several questions as described above. Once the child had worked through the entire interview for the first number of pirates (2), the interviewer then repeated the questions with a different number of pirates (for example, 3, then 4, or vice versa). Through analysis of these interviews, the EqPart LT was further specified into a more-or-less linear progression of levels, systematically moving through the cases. The interviews were intentionally open-ended: interviewers probed student reasoning when the students' responses or comments presented the opportunity, in order to take note of unexpected, original, or elaborated student behaviors and reasoning.

A critical development was the growing recognition of the importance of properties in explaining how students' cognitive reasoning progressed. While each of the cases required its own variety of strategies, across the cases particular properties concerning sets of tasks emerged, such as compensation (the recognition that increases/decreases in the number of persons sharing results in decreases/increases in share size), composition of splits (acting both vertically and horizontally on rectangles to produce multiplicative numbers of shares), transitivity (the recognition of equivalence of non-congruent fair shares generated from equal-sized wholes), distributivity of equipartitioning over breaking or fracturing of multiple wholes, etc. Developing proficiency in these "emergent properties" led to a re-design of the learning trajectory into a two-dimensional matrix.

Thus, early on in the development of the EqPart LT, the iterative process employed to refine it highlighted the need to differentiate between variations in the task parameters and the increased cognitive sophistication for the levels of the learning trajectory. By this phase of the work, it was clear that the learning trajectory could not be characterized as a simple linear progression of levels of behaviors. The EqPart LT was at this point represented in two dimensions: the "proficiency levels" of the concepts and behaviors on the vertical dimension, and the task parameters on the horizontal dimension. Not all task parameters were applicable to each concept level (for instance, "equipartitioning collections" only pertained to collections of discrete items, not whole dissectible shapes), nor was every concept or behavior relevant to every task parameter (for instance, composing splits on multiple wholes does not apply to single wholes).

In spring 2009, approximately 120 assessment items with individually narrowed focus, covering all relevant combinations of concepts or behavior level and task parameter, were composed and organized into 33 distinct overlapping forms for 4 grade bands: 10 forms of 6 items each for grades K-1 and 2-3, 8 forms of 8 items each for grades 4-5, and 5 forms of 8 items each for grades 6-7. Individual forms were designed for completion within a single class period, to minimally disrupt teachers' normal instructional schedules. The tests were administered to the entire student populations of 7 elementary and 3 middle schools in a rural, an urban, and two mixed suburban/rural North Carolina school districts. The research group examined all responses to each field-tested item in developing the rubrics. An example of an item and its associated rubric is shown in Figures 1 and 2.

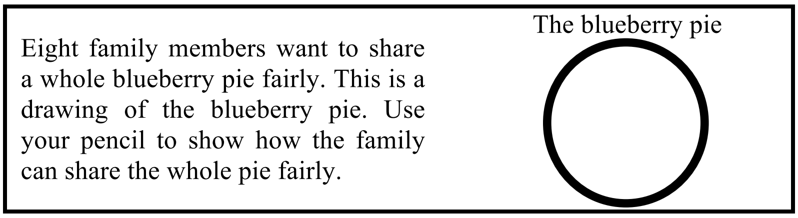
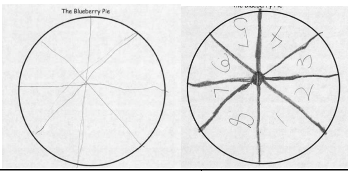
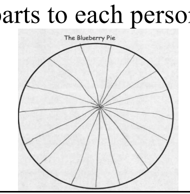
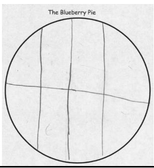
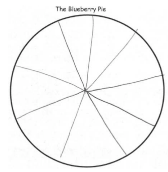


Figure 1. Item (Row 3, Task Class D) of Equipartitioning LT matrix
(size reduced from original)

<table>
<tr><th>Level</th><th colspan="3">Descriptions and Exemplars</th></tr>
<tr><td>2</td><td colspan="2">2A. Exhausts whole and creates 8 fair shares.
</td><td>2B. Creates 8k equal-sized parts and allocates k parts to each person.
Description:
Creates 16 equal sized parts and allocates 2 parts to each person.
[No exemplar in field test data]</td></tr>
<tr><td>1</td><td>1C. Creates 8k equal-sized parts but does not clearly allocate k parts to each person.
</td><td>1D. Creates 8 unequal-sized parts and exhausts whole.
</td><td>1E. Creates more/fewer than 8 equal-sized parts that exhaust the whole.
</td></tr>
</table>

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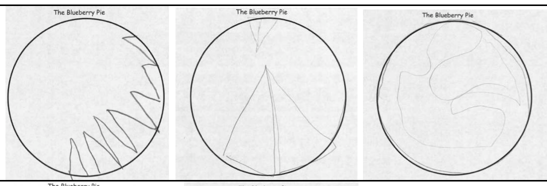
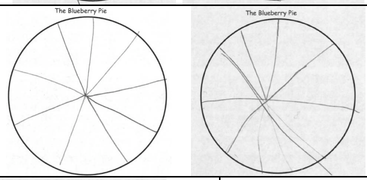
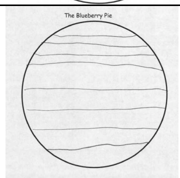
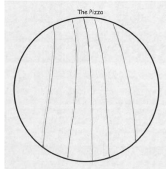
0	Incorrect or unintelligible.	
	0I. Insufficient evidence.	
	Special Code O: Creates *n+1* parts instead of *n* parts (# of cuts versus # of parts) (Here, makes 8 cuts (9 parts) when sharing among 8 people 	Special Code P: Creates parallel cuts to try to fairly split a circle (shape misconception). 

Figure 2. Rubric, based on student responses, for the item in Figure 1.

This phase moves the work closer to the aim of deploying diagnostic assessments aligned with the progression of concepts in the learning trajectory. Each item's responses are examined in detail to determine whether the item elicits a consistent and interpretable range of student responses that justifies its continued usefulness for inferring student cognitive understanding. Items needing revision to improve clarity or specificity are identified at this point, as are those that should be redesigned or even discarded.

Table 1. Equipartitioning Learning Trajectory Matrix. Proficiency levels form the vertical dimension, in the left column of the table. Task classes, listed along the top row, form the horizontal dimension.

Equipartitioning Learning Trajectory Matrix (Grades K-8)		**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
Task Classes→ / Proficiency Levels		Collections	2-split (Rect/Circle)	2^n split (Rect)	2^n split (Circle)	Even split (Rect)	Odd split (Rect)	Even split (Circle)	Odd split (Circle)	Arbitrary integer split	$p = n + 1$; $p = n-1$	p is odd, and $n = 2^i$	$p >> n$, p close to n	all p, all n (integers)
16	*Generalize*: a among $b = a/b$													
15	*Distributive property,* multiple wholes													
14	*Direct-, Inverse- and Co-variation*													
13	*Compositions* of splits, multiple wholes													
12	Equipartition *multiple wholes*													
11	Assert *Continuity principle*													
10	*Transitivity* arguments													
9	Redistribution of shares (*quantitative*)													
8	Factor-based changes (*quantitative*)													
7	*Compositions* of splits; *factor-pairs*													
6	*Qualitative compensation*													
5	*Re-assemble*: n times as much													
4	*Name* a share with respect to referent unit													
3	*Justify* the results of equipartitioning													
2	*Equipartition single wholes*													
1	*Equipartition Collections*													

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Item analysis and rubric development also serve a bidirectional role in the empirical process of refining and validating the learning trajectory. The outcome spaces for individual items help the research group determine whether individual items are consistent with and likely to be useful in inferring student progress with respect to the conceptual level from which the items were originally designed, and whether the item should be assigned to a different conceptual level. Conversely, and perhaps more important, the analysis of outcome spaces provides a form of test of the learning trajectory itself. The outcome spaces for different items of different cognitive levels are closely examined to ascertain whether the relative complexity and sophistication of reasoning evidenced by the students across items of different levels are consistent with the hypothesis represented by the levels of the EqPart LT, that of progression of student cognition from less sophisticated to more sophisticated.

Levels and Task Parameters of the Equipartitioning Learning Trajectory

The Equipartitioning Learning Trajectory is represented in a two-dimensional matrix (Table 1) comprised of the progression of proficiency levels along the vertical axis (listed from bottom to top), and the task classes along the horizontal axis (listed from left to right, roughly in order of case A to case C/D tasks). The proficiency levels and task classes are organized to accommodate the following behavioral and cognitive components of student reasoning for each equipartitioning case.

- The extent to which students successfully accomplish equipartitioning.
- Strategies and representations students use to accomplish the tasks, including multiple approaches
- Evidence of mathematical reasoning practices, including naming and justifying
- Emergent mathematical properties
- Predictable patterns of errors: misconceptions and critical barriers

Discussion

The analysis to date of clinical interviews and assessment item responses permit us to further support several major conclusions identified previously by Confrey and colleagues (Confrey 2008; Confrey et al. 2009):

- The EqPart LT is robust for children across the entire age range K-7. Older children in this age range, by and large, exhibit the same progression of reasoning about the tasks, strategies, and emergent properties as do younger children.
- Pothier and Sawada's (1983) conclusion, that number theoretic properties and geometric qualities are important in (equi)partitioning activities, is strongly supported.
- These foundations of rational number concepts do not require addition or subtraction operations, and can be developed in school in parallel to the development of those operations, rather than subsequent to them.
- Through Equipartitioning, (partitive) division cognitively precedes multiplication.
- Multiplication as re-assembly of equipartitioned collections or wholes can readily be conceptually developed instead of and earlier than it is typically developed as repeated addition, and that this would greatly strengthen children's subsequent multiplicative reasoning. We believe that deriving multiplication from repeated addition, as currently done in most curriculum, limits the development of children's multiplicative reasoning, constraining it inappropriately to additive reasoning, when it should be developed as a distinct but parallel conceptual reasoning field;
- Equipartitioning of multiple whole items leads in the higher levels of the EqPart LT to the distribution of equipartitioning over breaking or fracturing of sets or items, offering a foundation for the distributive property that can precede the numeric and algorithmic treatment of the property, and thus strengthen student flexibility with this critical concept that is essential for algebra, and which bridges additive and multiplicative reasoning.
- Behaviors, mathematical reasoning practices, and emergent mathematical properties that are elicited through equipartitioning of collections and continuous wholes, by children as young as 5 years old, anticipate the following rational number concepts:
  - Ratio, through two-dimensional 'many-to-one' numerical relationships, unit ratios, and ratio units;
  - Fraction-as-number, through the re-unitizing of fair shares to a many-as-one definition of the resulting fair share, unit fraction (students readily identify a single share of *n* fair shares as "$1/n^{th}$ of the whole), and possibly the partitive fraction quotient construct;
  - *a*/*b*-as-Operator, through the development of "$1/n^{th}$ of" and "*n* (times) as many" in naming fair shares and identifying the referent units for the fair shares resulting from equipartitioning.

Conclusions

The contributions of the current work include the definition and elaboration of a learning trajectory for Equipartitioning, a rational number construct that integrates previously disparate cases. This was done through a rigorous and systematic process of research synthesis, clinical interviews, and extensive analysis of student item responses from a large body of assessment items developed to align with the case-based analysis of fair sharing activities.

The development of assessment items and analysis of student item responses with respect to the learning trajectory levels is an empirical process that serves to validate the ordering and specific content of levels of the learning trajectory, on the one hand, and the opportunity to correct and re-align those levels with student responses.

Our work on learning trajectories in rational number reasoning represents a large-scale design problem. The work to date on the Equipartitioning learning trajectory provides a prototype for operationalizing the design of rational number learning trajectories as frameworks for instruction and assessment. Additional study of equipartitioning is underway, directed at development of corresponding diagnostic assessments. Further validation steps include the field testing of additional items, further elaboration of the EqPart LT proficiency levels, and the application of different statistical measurement models to prototype assessments.

A significant opportunity for further work is the construction of supplemental curriculum materials based on the Equipartitioning learning trajectory. We recall that we define learning trajectories as specifically involving an "ordered network of constructs a student encounters *through instruction*…" We note that Equipartitioning is not part of any curriculum of which we are aware. It is notable that older students who have not become proficient in equipartitioning exhibit misconceptions that could be remedied by systematic introduction to equipartitioning. This suggests that if equipartitioning were embedded in elementary mathematics curriculum, understanding of rational number could be accelerated in major ways.

Endnotes

(1) The process outlined here is informed by the "BEAR method" (Wilson 2005), adapted to link to our synthesis and interview protocols.

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