Fostering Mathematical Inquiry: Focus on Teacher's Interventions

Mara V. Martinez and Wenjuan Li
University of Illinois at Chicago

Abstract: Previous research has emphasized the need to better understand and articulate the demands in the work of teaching mathematics entailed by an inquiry-based approach. It is in this context that, first, we describe an inquiry task intended to provide high school students the opportunity to construct algebraic proofs. Second, as students work on the problem, we map students' inquiry process. More, we identify elements common to students' inquiry process and current views of how mathematics knowledge is constructed. Last, we illustrate the teacher's interventions intended to sustain students' inquiry. Among these, we identified: (1) helping students re-focus their inquiry, (2) helping students select mathematical tools, (3) accepting students' provisory ideas, (4) recognizing the potential in students' ideas and promoting the student to showcase the idea, and (5) reviewing a property using an additional example to preserve the original challenge for students.

Introduction

In the United States, the mathematics instruction that most students experience in today's classrooms embodies myths that misrepresent the nature of mathematics as well as what it means to learn and do mathematics. Mathematics textbooks, pedagogical practices, and current assessment policies work in concert to perpetuate the idea that mathematics is the discipline of certainty. This is despite current reform efforts as voiced in the NCTM Standards (National Council of Teachers of Mathematics, 2000) and as articulated in standards-based NSF-funded curricula. These documents advocate for a shift in emphasis from routine skills and factual knowledge to, posing and solving a variety of math related problems, reasoning and communicating mathematically, and appreciating the value of mathematics. Mathematical inquiry, also known as inquiry-based learning, discovery learning, or inquiry-based teaching, is an approach to teaching and learning that intends to engage students in authentic mathematical activity for the purpose of learning about what is entailed in doing mathematics. However, creating a classroom atmosphere and designing instruction to promote mathematical inquiry presents many challenges for the teacher (Goldman, Radinsky, Tozer, & Wink, in press; Lampert, 1995). For instance, individual students are interested in different questions or taking different approaches to the problems, but the teacher must teach the class as a whole. Or, students are inclined to use informal language or everyday life language in the inquiry process, but using formal language or precise disciplinary language is a major goal of learning in discipline. To facilitate inquiry learning, teachers need to know more about students' inquiry process and their thinking in different stages as well as effective scaffolding strategies that connect students' thinking with mathematics.

Therefore, the first goal of this paper is to describe an inquiry task that intends to provide high school students' an opportunity to learn about algebraic proof (Martinez, 2008). Second, to map students' inquiry process. And, last, to illustrate the teacher's intervention intended to sustain students' inquiry. Before addressing the goals, and to contextualize this study, it is necessary to revisit in more detail the nature of inquiry, mathematical inquiry, challenges associated with the implementation of and inquiry based approach, and research questions that still remain open in the field.

Framework

Inquiry

Inquiry based learning is a pedagogical approach largely embraced across disciplines and goes back in time as far as Socrates (Goldman, et al., in press) "Inquiry learning foregrounds the questions rather than the answer and places the focus on learning in the learner, not in the material that can be transmitted by the teacher" (Goldman, et al., in press). This is in opposition to the more predominant behaviorist view of learning that reduces learning to the acquisition of ready-made facts through listening, memorizing, and practicing. In an inquiry based learning approach, students in addition to learn the discipline, they would also learn habits of mind (e.g., form and pursue questions and the tendency to think critically, among others.)

In agreement with Lampert (1995), *mathematical inquiry* is an approach to curriculum and instruction that gives the teacher the responsibility for introducing content in a way that is illuminated and modified in response to students' questions and ways of thinking about that content. Specifically, the teacher defines the focus of inquiry by posing problems to the class while students take an active part in acquiring knowledge by generating not only answers, but formulations of problems, definitions of the terms of discourse, and analyses of alternative solutions. Such approach assumes a constructivist epistemology (e.g., von Glaserfeld, 1991) while at the same time integrating contributions from supporters of viewing mathematics as a humanistic discipline (e.g., Ernest, 1991). Moreover, it builds upon Dewey's and Peirce's view of knowledge as "a process of inquiry motivated by doubt" (Siegel & Borasi, 1994). Following Borasi (1994), mathematical inquiry calls for highlighting ambiguity and uncertainty in the mathematical content studied so as to generate genuine conflict or doubt, and, consequently, the need to pursue inquiry. Indeed, doubt and anomalies, namely something that contradicts our expectations, are considered motors for inquiry.

An inquiry stance has implications in the way that not only learning and teaching are conceptualized, but also mathematical knowledge (Siegel & Borasi, 1994). In fact, mathematical knowledge is fallible and created through a non-linear process in which the generation of hypothesis plays a key role; also, the production of mathematical knowledge is a social process and truth is constructed through rhetorical practices. For instance, as evidenced in Lakatos (1976), *once* the original theorem of Euler's on the characteristic of polyhedra was *proved*, it was challenged by several counter-examples yielding revisions on both premises of the theorem and definitions used in its proof. Clearly, this position is contrary to the view of mathematics as certain, objective, and unproblematic. Absolutist, Logicism, Formalist, Intuitionist and Platonist argue that mathematics is a body of absolute and certain knowledge; the truths of mathematics are universal; and mathematics is discovered but not invented (Ernest, 1998). In an inquiry epistemology, mathematics is regarded as a humanistic and cultural product; mathematics knowledge is fallible; and the results of mathematics are a changeable social product (Ernest, 1998).

Inquiry in the Mathematics Classroom

School mathematics is often presented as a body of absolute and certain knowledge. Learning mathematics in school is about knowing facts and procedures, and getting the right answers as quickly as possible (Lampert, 1990). Richards (1991) argued that the problems in school mathematics are habitual and unreflective and the discourse of school mathematics is just "number talk". On the contrary, mathematical inquiry is about encouraging students to ask mathematical questions; to solve mathematical problems that are new to them; to propose conjectures; and to be active participants in the construction of mathematical arguments (Richards, 1991). Central to the inquiry classroom is to offer students the opportunity to experience the uncertainty in the mathematical content, the nonlinear development in constructing mathematical knowledge, and to generate questions and to justify their answers (Siegel & Borasi, 1994).

Research centered on student thinking (Lampert, 1995) suggests that in this type of environment, the teacher's job should be a relatively simple matter of refining and connecting informal understanding with what we want students to learn in school. Research conducted on inquiry-based classrooms focusing on the work of teaching indicates otherwise.

Goldman et al. (in press) identify three challenges related to the classroom instructional context and the teacher's guidance. First, the teacher's level of preparedness to guide inquiry-learning projects; indeed, effective guidance of inquiry activities requires that teachers understand both the process and the content of the inquiry. Second, teachers need to shift norms of classroom discourse away from the traditional pattern (teacher initiation, student response, and teacher evaluation) to shared construction of reasoned arguments. Third, teachers need to move students' everyday language to a more formal language as it is used in the discipline.

Lampert (1995) identifies several tensions that a teacher navigates in an inquiry based mathematics classroom. For instance, individual students are interested in different questions, but the teacher must teach the class as a whole. Also, students' inquiry may push them deeply into one topic, but the teacher is responsible for their knowledge of a broad range of topics. Additionally, students' conjectures push them in different directions through the subject matter, but the teacher is responsible of keeping track of who knows what and what they still need to learn. Further, students may develop idiosyncratic systems for structuring their understanding, but they are also supposed to learn to communicate with a wider community that shares well-established conventions. In an inquiry-based classroom, teachers construct discourse in response to student activity, thus their responsibility is much grater. Therefore, teachers who want to teach in an inquiry based approach need to be able to navigate these tensions.

From a research perspective, little attention has been paid to the challenges of the teaching practice that must be addressed in this type of classroom environment, and how teachers navigate them. Specifically, less attention has been given to "how" to guide the inquiry process once students were engaged in it. It is in this context that this paper focuses on teacher's interventions intended to sustain students' inquiry process when conjecturing and producing an algebraic proof in a high school mathematics class.

Methodology

Teaching Experiment

The data was collected in the context of a teaching experiment conducted in a high school in the greater Boston in Massachusetts by the first author of this paper. One of the goals of the teaching experiment was to offer students the opportunity to conjecture in a context other than geometry[1], to use algebra as a tool to prove, and to produce algebraic proofs. It was intended to provide students the opportunity to experience proof as a way to understand "why" a mathematical phenomenon happens. In other words, in constructing the proof students would have the opportunity to access the reasons that make the mathematical statement true.

The teaching experiment consisted of a total of fifteen lessons. All lessons were video taped and all students' written work was collected. In addition, all small group conversations were audio taped. This allowed having an understanding of the class both at a macro level (whole group) and at a micro level (small group and individual students). Also, all students were interviewed individually twice; once half-way-through the teaching experiment and a second time at the end of the teaching experiment.

A group of nine students, who were in 9th and 10th grades, participated throughout the teaching experiment. Students worked in the same group of three for the duration of the fifteen lessons. Lessons were one hour long and were hold in addition to their regular mathematics class. During the fifteen lessons students worked on a set of seventeen *Calendar Algebra Problems* designed by the first author of this paper. The qualitative analysis[2] presented later is based on one of the three groups participating in the teaching experiment. In what follows, a description of the subset of data that is the focus of this paper is presented along with a detailed description of the problem that students solved during the first two lessons of the teaching experiment.

The Calendar Algebra Problem

As mentioned earlier, students worked on approximately 17 problems as part of their participation in the project. In this paper, data reported in this paper stems from students' work on the first Calendar Algebra Problem (Figure 1).

Problem 1

Part 1: Consider a square of two by two formed by the days of a certain month, as shown below. For example, a square of two by two can be

1	2
8	9

These squares will be called 2x2 calendar squares. Calculate the difference between the products of the numbers in the extremes of the diagonals.

Find the 2x2 calendar square that gives the biggest outcome. You may use any month of any year that you want.

Part 2: Show and explain why your conjecture is true always.

Figure 1. Problem 1 from the Calendar Algebra Problems.

Students were provided with calendars corresponding to years 2005-2008 accompanying Problem 1-Part 1 (Figure 1). As part of their work on Part 1, students had to analyze the nature of the outcome of the described calculation (subtraction of the cross product). It was expected that students would anticipate some kind of variation in the outcome in relation to the set of days where the operator is applied. This would contradict students' findings as a result of their exploration of the problem. The outcome is always -7 independently from where (i.e., within a month, across months, and across years) the square is located. This contradiction was intended to function as a *motor for inquiry* (Siegel and Borasi, 1994). The ultimate educational goal of Part 1 of Problem 1 was to get students to produce conjectures (correct and incorrect) about the behaviour of the outcome (i.e., number obtained as the result of the subtraction of the cross product) as it relates to the location of the square in the calendar.

After each group of students produced their conjectures, Part 2 of Problem 1, students had to gather evidence to show that their conjecture was indeed true. In addition, students had to figure out *why* this phenomenon happens, and whether this is *always* going to be the case.

Data Analysis

Data was analyzed qualitatively taking a grounded theory approach (Glaser & Strauss, 1967), which is a bottom-up approach. In other words, starting from the data theoretical relationships and categories are constructed. As mentioned above, in this paper, a map of students' inquiry process is generated based on the work on problem 1 of one group of three high school students. In addition, among the teacher's interventions we identified and analyzed the interventions that were intended to sustain students' inquiry process.

RESULTS

The non-linearity of students' inquiry process

Analysis of the group's discussions revealed the non-linearity of students' inquiry process. According to current epistemological perspectives (Ernest, 1998; Lakatos, 1976; Siegel & Borasi, 1994), mathematical knowledge is constructed through a complex process that involves several stages and includes the production of conjectures, examination of examples and counterexamples, arguments and counterarguments, reformulation of conjectures, redefinitions of terms, evaluation of theories, among others. This is in contrast with a more traditional epistemological view where the mathematician constructs a conjecture, proves the theorem, and as a consequence, truth is established forever. This is what we call linear process. It is characterized by a lack of loops going back to review or question prior accepted knowledge.

Indeed, students went through the following stages: Interpretation of the problem, conjecturing process, agreement on the conjecture to prove, determining how to prove the conjecture, proving the conjecture using algebra, and finally evaluating the implications of their work onto their mathematical knowledge.

Students started exploring the problem by using specific examples. Students will do this by placing the 2x2-calendar square in different places within a month, and in different months. As a result, they conjectured that the outcome would be always -7. This was in contradiction with their expectations. Initially, they had assumed that the outcome would vary depending on the location of the square. This has been described in the literature as follows "most often this feeling of doubt arises when an anomaly-something that doesn't make sense in light of existing beliefs- is encountered" (Siegel & Borasi, 1994, p.222). In this episode students encountered an anomaly, something that contradicts their expectations. This anomaly functions a motor for inquiry.

In what follows, students justified their conjecture stating: "it is -7 because there are 7 days in a week". Students are aware that this is not a proof; however, it seems that the fact that a week has seven days warrants the construction of their conjecture, and in a way, is reassuring. This is what Lakatos (1976) refer to as "conscious guessing" in the process of production of mathematical knowledge. Following Siegel and Borassi (1994), this "guess" enables to set in motion the process of knowledge production.

Once the proving process was set in motion, they seemed to agree that they needed to produce a general argument, probably using algebra. In doing so, students tried different paths, some of them correct and some incorrect. One of the incorrect attempts involved using a linear function when it was not the correct mathematical tool to use. At this point, the teacher stops by the group, and without questioning directly the correctness of the strategy, helps them to shift direction (Episode 1 in section below). The teacher's intervention helped them to make explicit and gravitate towards the idea of using variables so that they can continue working on it.

The next challenge that students faced was how many, and which independent variables to consider to have a complete algebraic model of the situation. First they used two independent variables to represent the numbers in the 2x2-calendar-square when, indeed one independent variable is needed. Once again, the teacher stopped by and helped students' to continue their inquiry process (Episode 2 in section below).

After that, students wrote the expression $a(a+8)-(a+1)(a+7)$ to represent in a general way the calculation performed on the 2x2-calmdar-square. Students were able to distribute transforming the expression $a(a+8)$ into a^2+8a. However, things did not go as well with the negative sign of the second term, namely $-(a+1)(a+7)$. At this point, the teacher intervened by recalling an example to illustrate how the negative sign would impact the expression (Episode 3 in section below). After overcoming this challenge, students constructed and algebraic proof showing why the outcome is always -7 independently of the location of the 2x2-calendar-square.

Students' inquiry describes a non-linear process, or what Lakatos called "a zig-zag" path. In this process, students conjectured, revised conjectures, gathered evidence, formulated and reformulated their ideas, devised strategies and re-evaluated them based on feedback provided either by the teacher, their findings, or peers. Students' inquiry process resembles current views of the production of mathematical knowledge. Contradiction functioned as a motor to engage students in inquiry. Students structured their work base on a "conscious guess" and produced mathematical knowledge upon it. In what follows, teacher's interventions are analyzed in the light of scaffolding or sustaining students' inquiry process.

Teacher's interventions intended to sustain students' inquiry

Teacher's interventions play a key role in sustaining students' inquiry process. However, little attention has been paid to the role of the teacher in sustaining students' inquiry once they are engaged in it (Chazan & Ball, 1999; Lampert, 1995). Given that teacher's discourse is constructed in response to student activity, the teacher's responsibilities and intellectual demands are substantial. The analysis yielded five teacher's interventions intended to sustain students' inquiry process. In what follows, these interventions are identified and illustrated.

Episode 1: Helping students to re-focus their inquiry

1 Student 2 (S2): It's … these the numbers increase at a constant rate right
2 S2: That's why it's always -7.
3 Teacher (T): So your hypothesis is that it doesn't matter where you place the square, you are going always to obtain minus 7, that is you hypothesis? Ok. So, how do you gather evidence to prove that? That is your problem now. How did you do? How did you come to that idea [outcome is always -7]?
4 Student 1 (S1): Did examples.

Interpretation: As part of their inquiry, students analyzed the underlying mathematical structure of the calendar, concluding that it was linear. Even though this is correct and potentially useful, the teacher interpreted that the students were focusing mostly on the writing of the linear relation. In addition, it seemed that students were aware of the link between the existence of a linear relationship and the fact that the outcome is always -7, as evidenced in the dialogue (lines 2 and 3). However, there was no evidence that students were analyzing *how* and *why* having a linear relation connects to their conjecture (i.e., the outcome is always -7). This indicated to the teacher that students probably thought that the mere existence of the linear relation was *sufficient* evidence to prove their conjecture. Therefore, the teacher brought back to the conversation the conjecture and the question "How do you gather evidence to prove that?" At the same time, the teacher wanted to link this last question to how they had arrived to their conjecture. This was an attempt to promote students to continue building on the examples they had used and, perhaps, construct specific linear relations among the elements of the 2x2-calendar-square. Note that the teacher here tries to sustain students' inquiry process by *returning* the question "how do we prove this?" and, by selecting a portion of their work (i.e., examples used to construct their conjecture) that had the potential to continue building the mathematical relations needed to prove.

Episode 2: Helping students select mathematical tools and accepting students' provisory ideas

1 T: Ok let's try something then. Can I write in this one? So, what we need to do is to gather information here… How can you show that for any number, this [difference between numbers that are the same column in the calendar and in consecutive weeks is 7] is going to be true?
2 S2: Oh I get it, so we do 'x'
3 T: Ok you can try.
4 S2: All right so like different variables.
5 S2: 'a' 'b' 'c' 'd', ok
6 S1: So, it doesn't matter, in a square is always …
7 T: Why did you come up with the idea of using letters?
8 S2: Using variables?
9 T: Yes, variables.
10 S2: Because if you have a formula that shows that they'll always be the same then it will work for anything so actually
11 T: Ok so try to work on that and see whether that gives us something solid to prove.

Interpretation: Until now, students had made great progress by finding out that in all the examples they tried the numbers in the first row of the square are exactly a week apart (i.e., seven days) from the corresponding numbers below (e.g., $\begin{matrix}2 & 3\\ 9 & 10\end{matrix} = \begin{matrix}2 & 3\\ 2+7 & 3+7\end{matrix}$). However, they were not sure how to represent the relationship mathematically capturing all cases, namely they were looking for a general expression. The teacher intervened by asking a question that included the phrase “for any number” (line 1) with the underlying hypothesis that this would help them to connect with algebra. As a result, one of the students suggested to use “x”, linking the problem to algebra. The teacher wanted to check whether they understood why “using x” would work (line 7). As evidenced in the student’s response “if you have a formula that shows that they'll always be the same then it will work for anything” algebra was being used in a meaningful way. The teacher intervention helped students to identify the mathematical tool that they needed by re-stating what they were trying to do and, deliberately, including the phrase “any number”. By helping students to identify a potentially useful mathematical tool, students were able to continue their inquiry process; therefore the teacher’s intervention sustained students’ inquiry.

In addition, note that when students suggested the use of four distinct independent variables “a, b, c, and d”, even though this idea could have been identified by the teacher “as incorrect or incomplete”, the teacher accepted it as a provisory idea. This intervention (accepting the idea) gave students the opportunity to students to further refine the idea by themselves as a result perhaps of their later work. More, students’ identification of 4 variables indicates progress in terms of their inquiry process; at this point, they do know that they are working with four variables. They still have to figure out that these four variables are indeed related among them (i.e., a, b=a+1, c=a+7, and d=a+8). This type of teacher intervention, accepting students’ provisory knowledge, helps sustain students’ inquiry process to that extent that gives students the opportunity to refine their “incomplete” ideas further.

Episode 3: Recognizing the potential in students’ ideas and promoting students to showcase their ideas

1 S2: I think that *a* and *b*… *b* is *a plus 1.* So if we make that 'a' plus 1 then we can cancel out both *a's* and that can be 1
2 T: Oh, did you hear his idea? Say it again please.
3 S2: All right so if we change 'b' to 'a' plus 1 cause that's what it's equal to then we can cancel 'a' and then just have 1 here
4 S3: I guess that'd work. Yeah.

Interpretation: One of the students in the group proposed the idea that *b* is indeed *a plus 1*. The teacher recognizing the potential of the idea in advancing students’ work, had him re-stating it for the other group members. Thus, this intervention, first recognizing the potential of the student’s idea, and second by showcasing it, helps students to advance their inquiry.

Episode 4: Reviewing a property using an additional example to preserve the original challenge for students

1 T: What is the impact of this minus in the signs of the elements here? Do you remember that?
…
2 T: Ok. […] Let's say that we have this example here -(3+x), if I want to make disappear the parentheses and this minus there do you remember what we do? Do we remember the rule to change the signs?
…
3 T: I will remind it to you, I will remind it to you, don't worry, don't worry. …
…
4 T: Yes? Ok so that is your challenge now. Here [pointing to the minus sign in -(x^2+8x+7)], how does this negative here impact the signs here? And, in that way you can get rid of the minus here and the parentheses and continue working in order to cancel,

which was your idea.
5 S3: Ok
6 T: Try that challenge now please.

Interpretation: Students did not remember how to distribute the minus sign in the expression "-(x^2+8x+7)". In order to help them, the teacher, created another example (i.e., -(x+3)) and remind them how the distributive property works. In doing so, she reminded them how to distribute the minus sign purposely using an expression different from "-(x^2+8x+7)". This intervention did not take away students' opportunity to apply the property once they remembered how it works. Once the teachers intervened, students knew how the property worked and still had the opportunity to do it by themselves.

In summary, the following five teacher interventions intended to sustain students' inquiry were identified: (1) helping students re-focus their inquiry, (2) helping students select mathematical tools, (3) accepting students' provisory ideas, (4) recognizing the potential in students' ideas and promoting the student to showcase the idea, and (5) reviewing a property using an additional example to preserve the original challenge for students. Employing these strategies, the teacher did not take away from students the intellectual challenge of the inquiry and provided support in response to and built upon students' activity. In doing so, the teacher's interventions sustained students' inquiry.

Conclusion

First, we provided a description of a task designed to foster students' mathematical inquiry. Specifically, it was intended to provide high school students' an occasion to learn about algebraic proof. The Calendar Algebra problem was designed to provide students the opportunity to experience a contradiction between, their expectations, and their findings as result of their exploration of the problem. Students' experienced this contradiction and functioned as a *motor for inquiry* (Siegel and Borasi, 1994). Second, we mapped students' inquiry process. Further, we identified elements of students' inquiry process that are described in current epistemological views of the nature of mathematical knowledge (e.g., non-fallible, it is produced through a non-linear process). Last, we identified and illustrated five teacher interventions (e.g., recognize the potential in students' ideas) that helped students sustain their inquiry. This is relevant to the field, given that little attention has been paid to how teachers guide students' inquiry once they are engaged in it (Chazan & Ball, 1999; Lampert, 1995). Still, we need to know more about how teachers construct and implement assignments that engage students in mathematical inquiry. We also need to know more about teacher's interventions that foster or hinder students' inquiry process. Once the field systematizes these results, they have the potential to inform the preparation of pre-service teachers and develop programs to better support in-service teachers to implement inquiry-based learning in mathematics.

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(1) In the United Sates, proofs appear mostly exclusively in the geometry curriculum in high school.

(2) The analysis was conducted in all groups participating in the teaching experiment. Given space restrictions we illustrate our findings with the analysis of one of the three groups. We chose this group given that the results that we wanted to illustrate appeared all as part of the work of this group.