Multiple solutions and their diverse justifications to the service of learning in early geometrical problem solving

Naomi Prusak, The Hebrew University, Jerusalem, Israel, inlrap12@netvision.net.il
Rina Hershkowitz, The Weizmann Institute of Science, Israel rina.hershkovitz@weizmann.ac.il
Baruch B. Schwarz, The Hebrew University, Jerusalem, Israel, msschwar@mscc.huji.ac.il

Abstract: The goal of this paper is to show that argumentation gains from being multimodal in learning geometry, especially at elementary levels. Grade 3 students participated in a year long course designed to foster mathematical reasoning. The course combined problem solving in dyads, peer argumentation and teacher-led discussions. We focus on one activity: identifying the types of solutions, the kinds of reasoning and the kinds of non-verbal actions (gestures, drawings, folding etc) used. We show how gestures and other non-verbal actions were interwoven with children's verbal peer argumentation and led them to new insights on the concept of area.

Introduction and Theoretical Framework

In recent decades, there has been a growing emphasis in mathematics education on fostering thinking and reasoning abilities. Leading researchers have pointed at special forms of talk that may support mathematical reasoning. Argumentative forms of talk have been favored by several researchers. In the realm of unguided small group talk, Schwarz and Asterhan (Asterhan & Schwarz, 2007, 2009; Schwarz & Linchevski, 2007) showed that argumentative talk may lead to conceptual learning and change. However, they also showed that it is not easily triggered and that several conditions (e.g., the presence of devices for testing hypotheses, or the timely introduction of specific argumentative scripts) are crucial for the emergence of productive argumentation (see also Howe et al, 2000). In mathematics education, teacher-led reflective talk and argumentative talk have been shown as crucial for the establishment of socio-mathematical norms (Cobb et al., 1997). Other researchers have studied conditions for unguided productive argumentative talk to show the role of principled design (Schwarz, Hershkowitz & Prusak, 2010). However, recently mathematical *problem solving* is recognized as based on multiple channels of communication – gestures, drawings, bodily actions, or manipulation of software. These channels of communication provide precious information on mathematics learning. As Radford (2009) claims: "Mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a genuine sense, by actions, gestures and other types of signs" (p. 112). Similarly, we show in this paper that the deployment of mathematical reasoning is often multi-channeled. In particular, we show the centrality of multi-modality in early geometry learning. More specifically, Duval (2006) has indirectly provided an interesting research direction. He linked argumentation in geometry to the *méréological decomposition* of shapes – divisions of the whole into parts with the aim of reconstructing another figure, which allows the detection of geometrical properties. This *méréological decomposition* can be done materially (by cutting and reassembling), graphically (by drawing lines that reorganize the shapes) and by looking. In our working hypothesis, we adopted Duval's theoretical idea that *méréological* decompositions involve productive argumentation. We will show that multimodality is a central characteristic of this argumentation and in general to early geometry problem solving.

Description of the experiment

The problem solving course

Three groups of 20 talented 3rd graders participated in a special enrichment program in mathematics over three successive years. The students attended 28 meetings over a whole academic year. The program was designed to develop *problem solving strategies* and practices such as drawing a diagram, adopting trial and error methods, identifying patterns and putting into action deductive considerations. About 25% of the activities dealt with issues related to the geometrical concepts of area and perimeter and their relationships. Typically, in each lesson (75 minutes long), the teacher first initiated a 15 *min*. long discussion to ground necessary understandings to be shared. Then, the teacher distributed worksheets, and encouraged students to work in dyads (up to 40-50 minutes). The teacher passed by the dyads to answer questions if needed. At the end of the activity, the teacher orchestrated a reflective discussion on the activity.

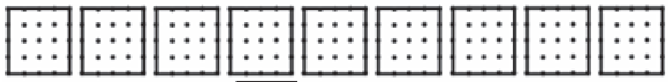
The activities were designed to trigger productive argumentation (Andriessen & Schwarz, 2009; Hadas, Hershkowitz & Schwarz, 2002). Three design principles were adopted: a) creating a conflict situation in which students were confronted with unexpected data or divergent opinions; b) encouraging participants to collaborate;

c) providing an environment for raising and checking hypotheses. The activity "sharing a cake" (Fig. 1) fulfilled these three design principles.

Sharing a cake

Yael, Nadav, and their friends, Itai and Michal came home from school very, very hungry. On the kitchen table was a nice square piece of cake, leftover from a birthday party. They wanted to be fair and wanted to divide the cake into **four equal pieces** so that everyone would get a fourth ($\frac{1}{4}$) of the leftover cake.

1) Suggest different ways in which the children can cut up and divide the square piece of cake. For each suggestion, explain why this would give each child exactly a fourth of the leftover cake.



2) Danny offers the following idea: 

Mindy immediately came out and claim: *your suggestion is wrong, don't you see, the parts cannot be equal!!* Who is right, and why? (Explain)

Figure 1. The first two tasks of the "Sharing a cake" activity

The goal of the first task is to encourage students to provide diverse solutions and diverse justifications. The students are explicitly required to explain and justify in writing each of the solutions they chose to draw. Nine grid squares representing the cake were given to students in their worksheets in order to encourage them to find many diverse solutions, and to provide a proper context for comparing areas of various shapes (especially non-congruent) created on the square grid. The goal of the second task is to trigger a cognitive *conflict* in which a non congruent partition of the square is presented. Moreover we introduced a text by the imaginary student Mindy to bring to the surface current misconceptions regarding the concept of area.

Methodology

The students' work and the classroom's work as a whole were observed, videotaped, transcribed and analyzed. We used four cameras: one directly on the teacher presenting the task; and conducting the whole class discourse at the end of most lessons. Two other cameras documented a dyad or triads throughout the whole meeting (75 minutes) and the fourth camera was mobile and documented small episodes of various interactions within small groups in the class.

Data analysis of the "sharing a cake" activity

Like in all the other meetings, students were asked to collaborate in pairs. They were asked to negotiate and reach agreement. A first analysis revealed that this task resulted in multiple and creative solutions. We identified four types of solutions (see Fig. 2). In Type A, all four shapes were congruent and were created by simple partitions: drawing diagonals, perpendicular bisectors or segments parallel to one side. 95% of the students proposed all three Type A solutions. In Type B, all four shapes were congruent but were created through more sophisticated partitions. For Type C, solutions consisted of two different pairs of congruent shapes. In Type D, there was no more than one pair of congruent shapes (all four shapes could be non-congruent – see Fig. 2). 100% displayed at least one Type A solution, 87% found at least one Type B solution, 71% found at least one Type C solution, and 60% of the students found at least one Type D solution.

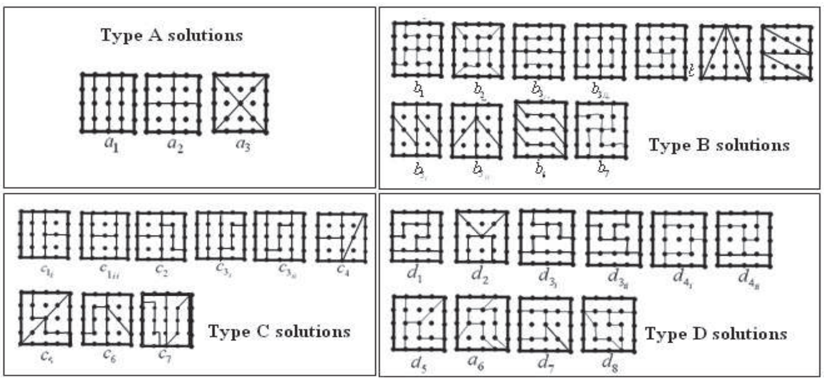


Figure 2. The four types of solutions and their subtypes

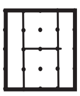
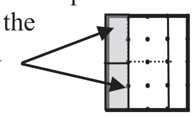
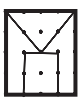
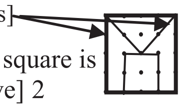
Table 1 shows that 84% of the students produced at least three types of solutions. In fact, they produced at least one solution in which not all of the four parts are congruent. Our goal to lead students to understand that not only congruent shapes have the same area was then achieved. In addition to the solutions drawn on the worksheets, scrutiny over the videotapes revealed that these drawings were the result of a rich interactive activity during which children justified their solutions and convinced their peers in various ways. In our analysis, we aimed at uncovering the actions and justifications involved in the solutions.

Table 1: The percentage of dyads that found more than one type of solution

Number of solutions	Type of solution	Percent	Total
Two types of solutions	A and B	8%	16%
	A and C	3%	
	A and D	5%	
Three types of solutions	A and B and C	29%	50%
	A and B and D	16%	
	A and C and D	5%	
Four types of solutions	A and B and C and D	34%	34%

Some examples are shown in Table 2. The first column refers to the type of solution peers elaborated on. The second column in Table 2 includes protocols of the verbal and non verbal action and solutions' justification. The third column includes the analysis of the justification deployed (in italics) and the non-verbal actions (underlined) used. Table 2 shows that for **each solution in the diagram, non-verbal actions are central to the reasoning process**.

Table 2: Reasoning examples accompanied co-constructions of solutions

Type of solution	Protocols of actions and justifications	Justifications' and actions' analysis
a_2 	Don: Each part is the same square The four parts are of the same sizes and with the same shape [Points with his finger along the division's lines] 	*Congruency justification* Drawing a virtual perpendicular bisector with finger to **explain solution**.
c_4 	Harry: This  we know is right and this also is right so  I divide the square in half and each half I divide in two equal parts, so I'm sure my solution is right 	*logical justification + Congruency* **Articulating deductive reasoning** verbally while pointing at a figure, to make clear the object of reasoning
c_{1ii} 	Lital: this and this these are equal[points to squares with her pencil], but the other two are different [points to the rectangles with her pencil] Ophir: but I know why these two are equal [pointing at a rectangle and a square]. One square is equal to one rectangle because it is" **as if**" it is divided and these two are equal to this. [Pointing with her pencil on the two parts of the divided rectangle, moving her figure along an imaginary line]  Lital: *Now I understand*	*Composing and decomposing* Transforming a figure (here decomposing or folding it) to **(counter)-challenge an argument in a logical way**
d_2 	Shay: *these two are the same, they have three squares* [Pointing with his pencil on the two congruent shapes]  And here and here this is two halves …now the square is 4 squares And this [pointing at the triangle above] 2 squares and 1/2 and 1/2 and 1/2 and 1/2 all together 4 squares.	*Counting Justification* Pointing at elements of a figure to support deductive reasoning (here through counting),

The first kind of justification we observed was *visual direct strategies* based on a holistic intuitive perception of congruent shapes. To justify his Type a_2 solution (see Table 2), Don declared: *The four parts have the same sizes and the same shape* (he meant congruent). He accompanies his verbal justification by a pointing gesture and by the virtual drawing of a segment. We called this type of justification *Congruency*. The second type we observed is the *composing and decomposing* justification. It consists of transforming one figure into a different one. The C_{1ii} type in Table 2 is a solution for which the composing and decomposing serves as a dialectical move to handle a disagreement. Ophir's gesture of moving of a figure around an imaginary line helps Lital being convinced by figuring out a transformation.

The third type we observed is *counting justification*. It consists of counting square units or dots. The counting is usually accompanied by a gesture of one-to-one pointing, as in Shay justified his solution d_2 : *these two are the same, they have three squares… And here and here this is two halves*. As shown in Table 2, Harry justified his solution c_4 by providing a logical argument.

We analyzed the distribution of written justifications to find that the most popular justification was based on congruency (49%), then counting (39%), leaving composing and decomposing to only 12% of the written justifications. We then examined relationships between the type of solution and the type of justification:
The ***Congruency*** justification dominates Type A and Type B solutions (about □0%). This is not surprising as for these types, all shapes are congruent. For Type C solutions, the most frequent kinds of justifications were ***compose and decompose*** and ***counting*** with around the same percentage (about 40% each). For Type D solutions, the most frequent kind of justification was the ***counting*** one (about 85%). It is reasonable to assume that the 15% who used compose and decompose for Type D solutions were led by the need to have congruent shapes. We found that a significant correlation between the solution type and its justification.
We investigated whether students' justifications belong to various types or they hold on one type. We found that more than □0% of the dyads use at least two types of justifications, and 14% of them used all three categories of justifications.
Task 1 helped students apprehend the problem of dividing a geometrical shape in parts whose areas are equal. However our hypothesis was that the compose/decompose strategy enables further investigation of the concept of area. But we found that only solutions whose type is B or C yield compose/decompose justifications. Task 2 (see Fig. 1).was designed to afford composing and decomposing. In the following section, we show for one dyad how Task 2 (1) afforded the emergence of Compose/Decompose strategies, (2) was accompanied by a rich array of non-verbal actions, and (3) led students to new insights upon the concept of area.

The case of Harry and Larry

In addition to the fact that argumentation in Task 2 may stem from the composition and decomposition of geometrical figures, Task 2 also provides an opportunity for creating a cognitive conflict; because many of the Grade 3 students identified equality of areas with congruency of shapes. In addition the text offered opposite alleged opinions held by two imaginary students, which was intended to lower the chances of avoiding facing the cognitive conflict. In fact, both opinions were plausible and deciding whether "Danny is right or wrong" was a real challenge that led the Grade 3 students to engage in argumentation. Harry and Larry, are in a socio-cognitive conflict: Harry claims first that Danny is wrong and Larry claims the opposite. We depict here the multimodal argumentation that led them to jointly reach the correct solution. Non-verbal actions are in brackets:

Larry 4: I think Danny is correct the four parts are equal
Harry 5: Only two parts [Points at 1 and 2 with two fingers, stares at drawing, then shows with two fingers that he measures the length of a first segment in part 2 and then measures the same length on the corresponding segment in part 1. Harry draws squares to divide each part to square units] (see Fig. 3).
Interviewer 6: Yes, so you claim that Danny is wrong?
Harry 7: There are here [pointing on partitioned squares he drew] (see Fig.3) more parts so this is larger [pointing with his finger moving around the boundary of part 3.]
Larry 8: But look, all these here [composes Parts 1 and 2] and here all this [composes Parts 3 and 4] is equal. All this is equal
Larry 9: because here it is half [composes parts 1 +2] and here it is also half.
[Larry brings a model of the cake he cut off from the appendix]
Larry 10: Because it seems to me like a half that way [makes an encompassing gesture on parts 1 + 2] and it is here...I [rotates the model of the cake and draws the following broken line]. (See figure 4) Larry cuts the model of the cake with scissors along the broken line to convince Harry that part (1 + 2) is equal to part (3 + 4)].
Harry 13: [looks carefully at Larry's efforts] I have an idea [Harry cuts part 4 and

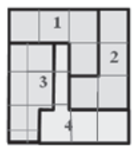


Figure 3: Harry's squares partition

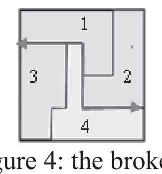


Figure 4: the broken line in Larry's drawing

reassemble it to look exactly like part 3] (See Fig. 5)
Larry 14: I told you they are the same. Danny is right.

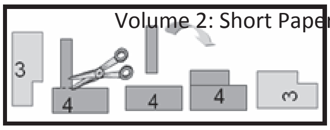


Figure 5: Harry's compose and decompose manipulation

A first analysis of this protocol helps showing that the composing and decomposing justification is central here (in Larry 8, 9, 10 and in Harry 13). It is involved in a rich argumentative activity which leads Harry to be convinced: Harry’s interventions in 5 and 7 are challenges to Larry in 4. Larry 8, 9 and 10 are counterarguments. The composing and decomposing justification is not fully verbal: Harry’s moving around the boundary of part 3 (in Harry 7) precedes Larry’s implementation of compose and decompose justification (in Larry 8); the encompassing gesture and the rotation in Larry 10 are ways to organi□e compose and decompose justification. The actions function as mental planning and monitoring of their strategies toward the solution of Task 2.

Some conclusions

The design of activities such as Task 1 afforded collaboration and experiencing problem solving processes which led to many solutions and to various types of justifications. And indeed the analysis of data in Task 1 shows that the young participants were challenged and produced many surprising solutions and justifications. The socio-cognitive conflict designed in Task 2 triggered the enactment of non-verbal actions that palliated the difficulty to articulate verbal justifications. With the help of these multiple channels, we observed new insights in the comprehension of geometrical areas – the fact that the shape of a geometrical figure can be changed without modifying its area. We also observed the seeds of deductive considerations as the compositions and decompositions were accompanied by verbal justifications including terms such as ‘because’ alongside with non-verbal actions. This *semiotic bundle* (Ar□arello, 2006; □adford, 2009) helped Harry and Larry to orchestrate reasoning in rich argumentative processes. We suggest that through these actions between the material (seeing, touching, and modifying) and the mental, children could function at an intermediate level to monitor and especially regulate their solutions.

References

Ar□arello, F. (2006). Semiosis as a multimodal process, □elime, Numero □special, 267-299.

Andriessen, □. □. B. □ Schwar□, B. B. (2009). Argumentative design. In N. □uller-□ir□a and A. N. Perret-□lermont (□ds.), *Argumentation and Education – Theoretical Foundations*. Springer □erlag.

Asterhan, □. S. □. □ Schwar□, B. B. (2007). The effects of monological and dialogical argumentation on concept learning in evolutionary theory. *Journal of Educational Psychology*, 99(3), 626-639.

Asterhan, □. S. □. □ Schwar□, B. B. (2009). Argumentation and explanation in conceptual change: Indications from protocol analyses of peer-to-peer dialogue. *Cognitive Science*, 33, 374-400.

□obb, P., Boufi, A., □c□lain, □. □ Whitenack, □ (1997). □eflective discourse and collective reflection. *Journal of Research in Mathematics Education, 28*(3), 258-277.

Duval, □. (2006). Les conditions cognitives de l'apprentissage de la g□om□trie: d□veloppement de la visualisation, diff□renciation des raisonnements et coordination de leur fonctionnement, *Annales de Didactique et de Sciences Cognitives, 10*, 5-53.

Goldin-□eadow, S. (2003). *Hearing gesture: How our hands help us think*. □ambridge, □A: Harvard □niversity Press.

Hadas, N., Hershkowit□, □., □ Schwar□, B. B. (2002). Between task design and students’ explanations in geometrical activities. *Canadian Journal of Research in Mathematics Education, 2,* 529-552

Howe, □., Tolmie, A., Duchak-Tanner, □., □ □attay, □. (2000). Hypothesis-testing in science: Group consensus and the acquisition of conceptual and procedural knowledge. *Learning & Instruction, 10*, 361-391.

□ercer, N., Wegerif, □., □ Dawes, L. (1999). □hildren's talk and the development of reasoning in the classroom. *British Educational Research Journal, 25*(1), 95-111.

□adford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meaning. *Educational Studies in Mathematics, 70*, 111-126

□e□nitskaya, A., Anderson, □. □., □ □uo, L. (2007). Teaching and learning argumentation. *Elementary School Journal, 107*(5), 449-472.

Schwar□, B. B., □ Linchevski, L. (2007). The role of task design and of argumentation in cognitive development during peer interaction. The case of proportional reasoning. *Learning and Instruction, 17*, 510-531.

Schwar□, B. B., Hershkowit□, □. □ Prusak, N. (2010). Argumentation and mathematics. In □. Howe □ □. Littleton (□ds.), *Educational Dialogues: Understanding and promoting productive interaction*. □outledge.