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Gestural Replays Support Mathematical Reasoning by Simulating Geometric Transformations

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Abstract: Actions relevant to conceptual ideas can promote thinking and learning, whether *internally generated* when spontaneously produced by learners (i.e., gestures), or *externally generated* when prompted to perform directed actions. Few studies have explored how prompting students to predict their future actions influences their thinking. This study compared the effects of internally generated predicted actions versus externally generated directed actions on undergraduate students' (N = 67) geometry proof performance. We investigated the role of *gestural replays* as physical re-enactments of one's prior actions involved in mathematical transformations. Gestural replays provide evidence of mental simulation processes of past actions. Quantitative models revealed significant benefits of gestural replays that depict students' prior mathematical transformations on proof performance, which extended to both externally generated directed actions and internally generated predicted actions. Qualitative analysis further illustrates ways gestural replays support embodied simulations that can bridge concrete actions and generalizations needed for mathematically valid proofs.

Introduction

Theories of grounded and embodied cognition (GEC) propose that cognitive processes are rooted in perceptual and motor systems (Wilson, 2002). One way mathematical ideas are grounded and embodied is through gestures. Teachers and students spontaneously use hand gestures to formulate their thoughts (Alibali & Nathan, 2012) and express their current and emerging understanding (Church & Goldin-Meadow, 1986). Gesture production generates new ideas (Novack & Goldin-Meadow, 2015), creates cohesion (Walkington et al., 2014), and promotes learning for a range of mathematical ideas (Goldin-Meadow et al., 2001; Smith, 2018).

Another line of evidence in support of GEC comes from studies showing that physical movements can be used intentionally to influence learning (Nathan, 2017). For example, students directed to move their eyes or arms (i.e., directed actions) in patterns that are compatible with the strategy for solving insight problems were more successful at generating correct solutions (Thomas & Lleras, 2007). Moreover, directing students to perform specific bodily actions tied to conceptual ideas but unrelated to a specific solution strategy can also facilitate the learning of abstract concepts in statistics (Zhang et al., 2021). However, Walkington and colleagues (2022) found that relevant directed actions by themselves did not directly improve high school students' (n=85) mathematical proof performance. Only when students' explanations included gestures of previous directed actions did the relevance of those actions become beneficial. These mixed results raise an important question: **How do *specific* body-based processes influence *specific* mathematical reasoning**?

We explore this question in the context of geometry proof, where participants were asked to either *perform* directed actions or *predict* actions that simulate geometric transformations as they reason about universal claims regarding the nature of space and shape. This study investigates the effect on proofs of students' spontaneous *gestural replays* -- physical re-enactments made during their explanations of one's prior actions of mathematical transformations (Beilock & Goldin-Meadow, 2010; Walkington et al., 2022). We report quantitative and qualitative analyses that examine the role of gestural replays and illustrate the ways that gestural replays can support participants' geometric reasoning and broader aspects of mathematical cognition.

Theoretical framework

Assumptions about cognitive processes

This work builds upon some assumptions about cognitive processes. First, cognitive processes operate within a predictive architecture. Rather than passively waiting for input to think, people continually anticipate what is to come in streams of sensory input and are poised to proactively respond (Clark, 2015). This predictive stance orients people to engage in sensorimotor simulations that project how one's behaviors will change the world and how the world will change in response to these behaviors.

Second, cognition is grounded and embodied. Barsalou (2008) proposed that meaning derives from perceptual and motor experiences from interactions with the world. This model also holds for *offline cognition,* when task-relevant inputs and outputs are no longer physically present (Barsalou, 2008; Wilson, 2002). Many of these cognitive tasks are accomplished through mental *simulations* of actions. For example, in language comprehension, reading action words (e.g., *kick*) activates neural pathways of the relevant sensorimotor systems (e.g., leg); readers automatically simulate these actions stated in words and ground arbitrary symbols and sounds to their cultural meaning (Glenberg & Gallese, 2012). Similarly, during mathematical reasoning, many mathematical concepts and symbolic systems of notation gain meaning by being grounded in perceptual systems and actions, including gestures and movement (Abrahamson & Sánchez-García, 2016; Alibali & Nathan, 2012).

Gestures as simulated actions for fostering mathematical reasoning

Spontaneous gestures have been shown to predict conceptual reasoning and learning by contributing to different types of mathematical reasoning (e.g., Cook & Goldin-Meadow, 2006; Ottmar & Landy, 2017; Smith et al., 2014). For example, children encouraged to gesture while explaining their solutions to mathematics equivalence problems were more likely to express new and correct problem-solving strategies compared to those told not to gesture and those told to explain their solutions with no mention of gestures (Broaders et al., 2007). Restricting gestures impairs model-based inference-making but not fact retrieval (Nathan & Martinez, 2015).

Some gestures arise from mental simulations of actions or perceptual states (Hostetter & Alibali, 2008). As simulated actions, gestures can highlight the spatial-temporal information from actions (Beilock & Goldin-Meadow, 2010) and help schematize relevant information to facilitate encoding (e.g., So et al., 2014) and generalization (e.g., Novack et al., 2014) of cognitive processes. Specifically, *dynamic depictive gesture*s (Garcia & Infante, 2012) enact spatial-temporal transformations of mathematical entities. For example, merely tracing a triangle highlights its static properties, whereas a dynamic depictive gesture can portray the invariance of the sum of its interior angles by re-scaling (transforming) its size, thus supporting generalization and abstraction. By simulating geometric objects and transformations, dynamic depictive gesture production predicts the formation of generalized mathematical proofs (e.g., Nathan et al., 2021; Pier et al., 2019).

Directed actions affect gesture production and non-verbal cognition

While people's movements can reveal their thinking, directing them to perform directed actions as part of an intervention also affects thinking and learning. Some studies have shown that directed actions from earlier training leave a legacy in gesture production in subsequent performance (Donovan et al., 2014). Cook and Goldin-Meadow (2006) found that children were more likely to produce gestures when given instructions that included actions about a solution strategy. Moreover, children's gestures were "picking up on, and reproducing, the content of the instructor's gesture" (p. 217). Performing gestures, in turn, led to better problem-solving performance on a post-test compared to children who expressed a solution strategy in speech only.

When relevant to conceptual ideas, performing directed actions, even without conscious awareness of their relevance, can facilitate learners' performance (e.g., Thomas & Lleras, 2007; Zhang et al., 2021). For example, Nathan and colleagues (2014) experimentally investigated the influence of performing directed actions on undergraduate students' proof performances. All participants were directed to perform either *task-relevant* actions or *task-irrelevant* actions prior to tasks. Task-relevant actions embodied the conceptual relations that underlay the proof tasks, while task-irrelevant actions were matched using the same number of steps and touch points but did not embody the same conceptual relations. Participants who performed task-relevant actions were more likely to generate correct intuitions (i.e., snap judgments) and key mathematical insights for subsequent tasks than those who performed task-irrelevant actions. This finding suggests that relevant directed actions facilitate intuitive and nonverbal processes. However, performing directed actions alone was ineffective for generating valid proofs that required students to consciously describe their chain of reasoning.

Gestural replays moderate cognitive relevance of directed actions

Walkington and colleagues (2022) also examined the influence of performing relevant directed actions on geometric reasoning in the context of an embodied video game that tracked high school students' movements. They observed that some of the explanatory gestures made by players were actually "replays" of the directed actions that were elicited during game play. These replays could be exact copies -- as when players' crossed arms matched the crossed arm movements they performed in the game -- or recreations of the same relations using different body parts, such as crossed hands or fingers. The investigators found that although performing relevant directed actions did not *directly* cause learners to produce more gestures or improve their performances, participants who produced gesture replays of previous cognitively relevant actions during their explanations showed significantly better insight and higher proof performance. Moreover, this effect was most consistent for insight and proof when those replays were dynamic depictive gestures. These findings suggest that the presence of gestural replays derived from relevant directed actions *moderated* the effect of those actions on proof performance. These gestural replays appeared to support embodied simulation by bridging the concrete actions performed in gameplay to the generalized reasoning used to establish mathematically valid geometry proofs. Walkington and colleagues concluded “that engaging in these ‘gestural replays’ of their actions during explanations--even when those replays were not identical recreations of the original actions--changed participants’ encoding for the mathematical principles of the task and subsequent task performance” (p.27).

Research question

In prior research (Xia et al., 2022), we found that prompting participants to predict possible actions for geometry conjectures produced a marginally significant effect on the generation of valid proofs (Cohen’s $d = .25$, $p = 0.07$) compared to actually performing the relevant directed actions. Moreover, neither condition led to more gestures, suggesting that gesture is not a mediator. We and others (e.g., Walkington et al., 2022) also observed the prevalence of gestural replays during students’ explanations. In light of these findings, we revisit the data post hoc to investigate: **How do specific body-based processes influence specific mathematical reasoning**? We hypothesize that gestural replays of directed or predicted actions made during students’ explanations provide a bridge between these actions and generalized mathematical reasoning that can enhance proof performance. We explore this hypothesis using both quantitative and qualitative analyses.

Methods

Participants & procedure

Participants were undergraduate students from a large university in the Midwestern US who participated in a 2X2 design comparing Directed Action Yes (DA) or No (DA') and Predicted Action Yes (PA) or No (PA'). For this secondary analysis, we focus on the two diagonal cells: DA+PA' (directed actions without predicted actions; n = 30) and DA'+PA (predicted actions without directed actions; n = 37) because it would be impossible to tell the source of the gestural replays in the DA+PA group, and there are no replays in the DA'+PA' group.

All participants were prompted to individually read each of the eight geometry conjectures, statements that are false or always true, with order varied by a Latin Square. DA+PA' participants were directed to mimic relevant actions comprised of a sequence of three animated poses without any prompt to make predictions (see an example in Figure 1). DA'+PA participants saw no directed actions and were prompted to predict the pose sequence for each conjecture (see an example in Figure 2). Participants completed each conjecture task by judging the conjecture's veracity (i.e., false or always true) and providing a verbal justification. Finally, each participant was asked to complete surveys about demographics, math history, and spatial skills.

Coding

Proof validity

Video recordings of participants’ explanations for each conjecture were transcribed and coded (0/1) for mathematically valid proofs (reliability $\kappa = .96$) based on Harel and Sowder’s (2005) characteristics of valid deductive proofs that must be simultaneously logical, operational, and generalizable.

Gestural replays

Spontaneous gestures made during mathematical explanations were first coded as *gestural replays* or not (1/0). Gestural replays were further coded at two levels: (1) exact replays exactly matched the directed (DA) or predicted actions (PA); (2) corresponding replays matched DA or PA to different body parts (e.g., DA crossed arms could match crossed hands in the explanation). Gestural replays coded were further classified as either *non-dynamic* or *dynamic* depictive gestures (0/1), as described earlier. Inter-rater reliability is forthcoming.

Results

Quantitative analysis

We ran mixed-effects logistic regression models (Snijders & Bosker, 2011) to predict participants’ proof performance. Our earlier analyses showed that *dynamic gestural replays*, either of DA or PA, were significantly associated with proof performance; while performing DA or PA did not cause more dynamic gestural replays, suggesting that dynamic gestural replay is not a mediator. Our current analysis explores how gestural replays influence the effects that DA and PA have on proof performance, using a model with an interaction between *dynamic gestural replay* and *experimental condition* (Walkington et al., 2022). *Participant ID* and *conjecture* were included as random effects. *Students' most advanced previous math course* and *spatial thinking* were retained in the model as covariates that significantly reduced the deviance of our model. We report odds ratios that are exponentiated raw coefficients and effect sizes using d-type measures (Chinn, 2000).

$$\begin{aligned}&logit(P_{ij}) = \gamma_{00} + \gamma_1 \times (experimental\ condition) + \gamma_2 \times (dynamic\ gestural\ replay)\\ &+ \gamma_3 \times (experimental\ condition) \times (dynamic\ gestural\ replay) + \gamma_4 \times (math\ course) +\\ &\gamma_5 \times (spatial\ thinking) + U_{0j\ participant} + T_{0j\ conjecture} + \boldsymbol{\varepsilon}_{ij}\end{aligned}$$

The results show that for DA+PA' trials, making dynamic gestural replays had a significant effect on generating mathematically valid proofs (*OR* = 2.76; *d* = 0.56, *p* = .006). For DA'+PA participants, making dynamic gestural replays also had a significant effect on proof performance (*OR* = 8.59, *d* = 1.19, *p* < .001). In contrast, for trials where students did not produce gestural replays, performing DA or PA was not significantly associated with proof performance. In trials where participants did perform dynamic gestural replays, DA'+PA participants significantly outperformed DA+PA' (*OR* = 3.11, *d* = .63, *p* = .019), even when controlling for math education and spatial ability. Gestural replays appear to strongly influence proof performance.

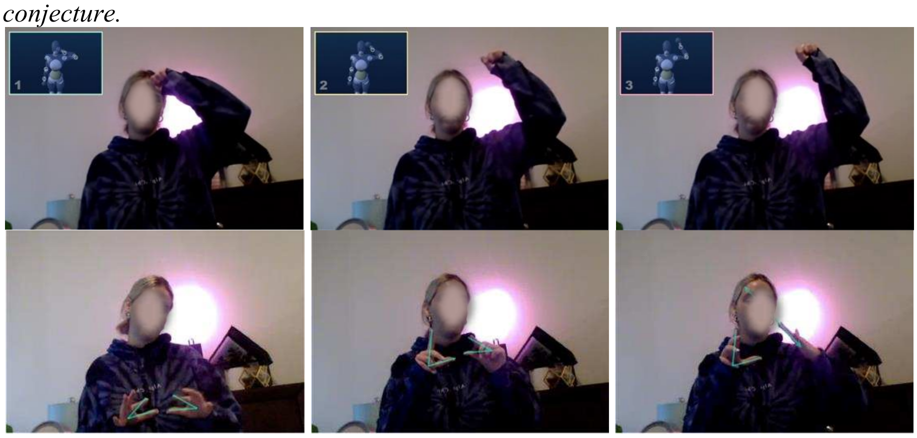
Qualitative analysis

To gain additional insight into how gestural replays might bridge actions to generalized mathematical reasoning, we qualitatively examined two cases. One participant produced gestural replays of directed actions that were *externally generated* (i.e., designed by researchers); the other produced gestural replays of predicted actions that were *internally generated* by the participant.

Case 1: Student's gestural replays from externally generated directed actions

We focus on this student's (S1; Figure 1) reasoning process after they have mimicked the relevant directed actions designed for the conjecture, *In triangle ABC, if Angle A is larger than Angle B, then the side opposite Angle A is longer than the side opposite Angle B*.

Figure 1
(Top Row) *Student performing externally generated directed actions shown in the picture inset.* (Bottom Row) *Student replaying modified versions of directed actions while evaluating the conjecture.*



(a) **(b)** **(c)**

Transcript #1 of S1:

[1] ((*Reading the on screen text*)) In triangle ABC, if Angle A is larger than Angle B, then the opposite side, the side opposite Angle A is longer than the side opposite Angle B, um, ((*performing movements for 27 sec*)), true.
[2] ((*Reading the on screen text*)) Explain why the statement is always true or false.
[3] Because if Angle A is opened up more than Angle B and it's bigger, doesn't even matter what side, then that line opposite would span a longer distance than would the line of Angle B.

Figure 1 (Top Row) includes an inset (upper left corner) that shows how participants were directed to mimic the on-screen avatar by raising their left forearm, bent at the elbow, to open up the angle created by the upper and lower portions of their arm. This sequence of motions was designed to highlight a key relation: as one angle of a triangle opens up, the side opposite the angle necessarily becomes larger.

After performing the directed actions, S1 contemplated aloud "if Angle A is larger than Angle B" (Line 1) while gesturing with the thumb and index finger of her right hand to form Angle A. Next, she compared this angle to a smaller Angle B that she formed using the thumb and index finger of her left hand (Figure 1a; Bottom Row). Juxtapositioning these two angles, the student moved the thumb and index of the right hand to open (increase) and close (decrease) Angle A while gazing at the changing angles. This illustrates a transformation of directed actions by juxtaposing the actions (i.e., angles) that originally occurred sequentially (Nemirovsky & Ferrara, 2009) and changing an angle by expanding and contracting. This process helps the student ascertain that the mathematical conjecture is true.

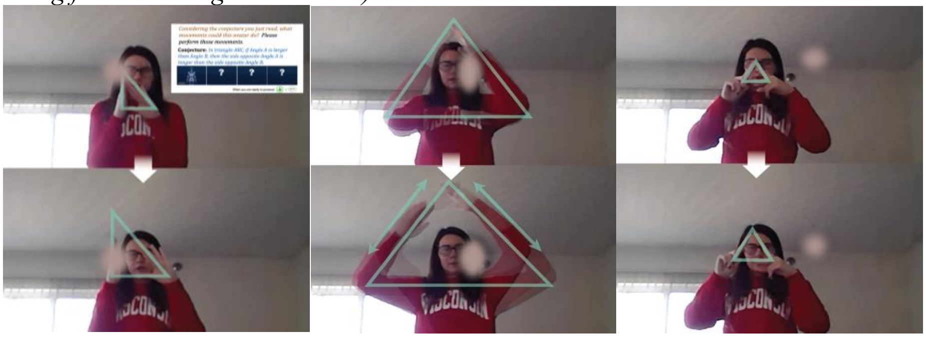
In her follow-up explanation (Line 2), S1 continued to justify that the conjecture is true: "... if Angle A is opened up more than Angle B and it's bigger" while holding the pose of Angle B and moving her right index finger up to make Angle A bigger (Figure 1b). Although the student performed a modified version of the original directed actions (i.e., switching from arms to hands and fingers), her *dynamic depictive gestures* constitute "corresponding gestural replays" that glean the relevant perceptual-motor information from the directed actions, which embody the key mathematical relation. She continued that it "doesn't even matter what side," suggesting that she was generalizing the mathematical relationship between angle size and side length for all triangles -- not just this particular example. In this way, the gestural replay may serve as a bridge between directed actions and generalized reasoning. In a second dynamic depictive gesture (Figure 1c), S1 moved her left hand back and forth to depict a "line opposing" the angle (Line 3), reinforcing that a larger angle always produces a longer opposing side length.

Case 2: Gestural replays from internally generated predicted actions

In this case, S2 read the following conjecture: "*For a triangle that is similar to triangle ABC, the side opposite to angle B must have the same length*" (a false conjecture). Then, S2 was prompted to predict possible actions for the conjecture but given no instructions on what movements to predict (see the inset in Figure 2a). Thus, unlike the DA condition, the movements in the PA condition were internally generated.

Figure 2

*Starting (*top*) and ending (*bottom*) poses of three different sets of predictions using (a) hands, (b) arms, and (c) fingers (with graphic overlays of the geometric objects being formed through movements).*



(a) (b) (c)

When asked to predict a series of movements for the conjecture, the student created a few different options, each building off the previous ones. For the first sequence (Figure 2a, Top), S2 created a triangle with her left palm and the fingers of her right hand. Then (Figure 2a, Bottom), S2 created a second, similar triangle by spreading her hands apart horizontally and using a pointing gesture as she described, "extend[ing] my fingers out" to complete the triangle. For the second set of predicted actions (Figure 2b, Top & Bottom), S2 made a set of similar triangles using her arms and moved both arms outward to make the triangle bigger with the same angles. In her third and final sequence (Figure 2c, Top), S2 used three fingers to first represent a triangle and then (Figure 2c, Bottom) made the triangle sides longer by moving her fingers outward while trying to keep the angles the same. Across all three predicted action sequences (Figure 2a, 2b, 2c), S2 made *dynamic depictive gestures* to demonstrate geometric transformations of geometric objects that explored general properties of triangles. We also noticed S2's representations ultimately refined to smaller body movements (i.e., fingers).

The reasoning also progressed to a generalized proof. After correctly identifying the conjecture as false (Transcript #2, Line 7), S2 explained "because although the angles will stay the same, the side lengths will change…" (Line 8). She then added that "the side length has to get bigger" (Line 9) while her gestures quickly shifted from gross-motor usage of the forearms to using three fingers to represent all three sides of the triangle. She then shifted from a self-oriented process of ascertaining, to a social-oriented process of persuasion directed at the researcher, saying that "if you're extending it out but want to keep the angles the same, the side lengths have to change" (Line 9). While speaking, S2 spread out her fingers to make the side lengths longer. This *dynamic depictive gesture* sequence is an "exact gestural replay" that exactly resembles the third representation of S2's internally generated predicted actions (compare Figure 2c to Bottom Row of Figure 3). Upon analysis, we see that S2's verbal explanations explicitly said, "side lengths [can] change" while replaying the predicted actions. However, her speech in the prediction phase focused on "similar triangles" and "same angles," though her movements embodied the idea that the side lengths can change. This suggests that the gestural replay helps bring the nonverbal forms of knowledge into verbally coded awareness, and that the combination of nonverbal, intuitive forms and verbal forms of knowledge leads to a valid, generalized mathematical proof (Nathan, 2017).

Figure 3
(Top Row) *Starting, Intermediate, and Ending poses of the third set of predicted actions from Figure 2(c).* (Bottom Row) *Student replayed the predicted actions during verbal justifications.*

Transcript #2:

[1-6] … ((*S2 first reads the conjecture and then predicts actions while speaking rationale*))
[7] This statement is false.
[8] This statement is false because although the angles will stay the same, the side lengths will change, that's what makes triangles similar.
[9] The side length has to get bigger because if you're extending it out but want to keep the angles the same, the side lengths have to change.

Discussion

Embodied interactions offer promising pathways for improving mathematical thinking and learning. As these approaches proliferate (Abrahamson & Trninic, 2015; Nathan & Walkington, 2017; Ottmar & Landy, 2017; Smith, 2018) -- in classroom curricula, video games, for example -- scholars in the learning sciences need to move beyond assertions that embodiment either *does* or *does not* improve learning and address a theoretical need to identify *how* these interactions recruit body-based resources as part of a broader understanding of embodied learning, and a practical need to identify *when* embodied interactions benefit mathematical reasoning. In answer to the research question, we found evidence supporting the claim that gestural replays provide a kind of "bridge" between actions and verbalizable conceptual reasoning that is critical for articulating mathematically valid proofs in geometry. This applied to both externally generated directed actions (DA+PA') and to internally generated predicted actions (DA'+PA). This investigation contributes to a growing body of empirical research showing that it is not mere *movement* but the type of movement and its conceptual meaning that is most consequential (e.g., Walkington et al., 2022; Zhang et al., 2021).

In geometry proofs, students must offer arguments that are logical, operational, and generalizable in order to defend universal claims about properties of space and shape. To achieve this, interlocutors rely upon physically enacted simulations of transformations of imagined objects in the form of *dynamic gesture replays*. Each term in this phrase has a theoretical import. That they are *gestures* indicates that students' conceptualizations of the relevant ideas are body-based, challenging traditional notions that privilege symbolic and verbal accounts of knowledge and abstractions and offering instead a more distributed account that extends notions of cognition to encompass non-symbolic body movements and the sensorimotor processes associated with these nonverbal behaviors. That they are *dynamic* indicates the power of enacting transformations on imagined mathematical objects to support the type of generalized thinking that is often attributed to abstract formalisms (and often regarded in contrast to concrete experiences). That they are *replays* indicates that students reinvoke these behaviors as a valuable cognitive resource, often preserving the original relational information even as they are gesturally "revoiced" using different body parts.

This study also explored the influences on embodied mathematical reasoning in terms of the history of these movements. Many embodied curricula and game-based interventions use externally generated movements to bring about the desired behaviors, prompting students to mimic the actions of another, touch certain locations, or follow certain patterns (e.g., Cook et al., 2006; Nathan et al., 2014; Thomas & Lleras, 2007). The present study is one of the very few that compare the effects of performing externally generated directed actions to performing internally generated predicted actions. The results are notable: When producing dynamic gestural replays, participants who predicted actions showed superior proof performance compared to those who mimicked cognitively relevant directed actions that had previously been shown to be advantageous to proof performance. Enactivist theoretical accounts (e.g., Abrahamson & Sánchez-García, 2016; O'Regan, & Noë, 2001) are likely involved, as these hypothesize the ways that sensorimotor behaviors such as dynamic gestures emerge as a solution to an interaction problem, influenced by feedback from the environment (including one's own actions) that guide future actions (including gestural replays). For example, students may come to experience an appropriate geometric transformation because of the actions they perform (e.g., expanding one's hands away from one another while they each form the shape of a vertex between the thumb and fingers) rather than acting on a priori images of geometric objects (e.g., dilating a triangle).

The qualitative analyses illustrate how gestural replays can bridge concrete actions and generalized geometric reasoning. For example, S1 replayed the externally generated directed actions, thus re-enacting the movements into personally meaningful gestures. S2's internally generated predicted actions engaged body-based resources to simulate the possible solutions and operationalize her own geometric thinking, which may have also helped to bring S2's nonverbal ways of reasoning into conscious awareness (Aleven & Koedinger, 2002), leading to a multimodal explanation that served as a mathematically valid proof of her claim. In each case, gestural replays simulate and schematize generalizable mathematical relations embodied by the relevant actions to enact mathematical transformations supporting generalizable geometric reasoning.

These findings need to be replicated across different populations and tasks. Still, this study offers two insights for the design of embodied learning interventions. First, there is value in directing students to perform relevant actions, but explicit prompting for them to replay those directed actions through personally meaningful gestures that enact mathematical transformations strengthens the influences of directed actions for proof. Second, simply encouraging students to predict actions for corresponding tasks may benefit learning, expanding the forms of embodied interventions that can benefit thinking and learning.

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